

FAILURE OF NECESSITY OF THE ENERGY CONDITION

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ABSTRACT. We give an example of a pair of weights $(\hat{\sigma}, \hat{\omega})$ on the line, and an elliptic singular integral operator H_b on the line, such that $H_{b,\sigma}$ is bounded from $L^2(\hat{\sigma})$ to $L^2(\hat{\omega})$, yet the measure pair $(\hat{\sigma}, \hat{\omega})$ fails to satisfy one of the energy conditions. The convolution kernel $K_b(x)$ of the operator H_b is a smooth flattened version of the Hilbert transform kernel $K(x) = \frac{1}{x}$ that satisfies ellipticity $|K_b(x)| \gtrsim \frac{1}{|x|}$, but not gradient ellipticity $|K'_b(x)| \gtrsim \frac{1}{|x|^2}$. Indeed the kernel has flat spots where $K'_b(x) = 0$ on a family of intervals, but $K'_b(x)$ is otherwise negative on $\mathbb{R} \setminus \{0\}$. On the other hand, if a one-dimensional kernel $K(x, y)$ is both elliptic and gradient elliptic, then the energy conditions are necessary, and so by our theorem in [SaShUr6], the $T1$ theorem holds for such kernels on the line.

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1. INTRODUCTION

In our main result, Theorem 1 below, we provide a counterexample to the necessity of the energy condition for the boundedness of a small, but elliptic, perturbation of the Hilbert transform. This answers in the negative Problem 7 in [SaShUr7] and Conjecture 3 in both [SaShUr8] and [SaShUr9], which if true would have established the celebrated $T1$ theorem for such operators - namely that boundedness of T is equivalent to a Muckenhoupt condition and testing the operator and its dual on indicators of intervals. In fact, the only known proofs of two weight $T1$ theorems to date all rely crucially on the necessity of the energy condition. On the other hand, in a relatively simple adaptation of an existing argument in [LaSaUr2], together with our main theorem in [SaShUr9] (see also [SaShUr6] for a more leisurely exposition), we show in Theorem 2

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below, that for a class of singular integral operators on the line that narrowly avoid our counterexample (i.e. their kernels are both elliptic and gradient elliptic), the $T1$ theorem does indeed hold for these operators.

Conjecture 1. *The energy conditions (see [SaShUr7]) are necessary for boundedness of a vector of standard singular integrals in higher dimensions provided the vector singular integral is both strongly elliptic (see [SaShUr7]) and strongly gradient elliptic. If this conjecture is true, the $T1$ theorem would then follow for such operators by the main theorem in [SaShUr9] or [SaShUr6].*

In order to better explain the significance of these results, we briefly review the history of weighted norm inequalities for the Hilbert transform. For a signed measure ν on \mathbb{R} define

$$(1.1) \quad H\nu(x) \equiv p.v. \int \frac{1}{x-y} \nu(dy),$$

for an appropriate truncation of the kernel $\frac{1}{x-y}$ - see e.g. [SaShUr7] or [SaShUr6]. A *weight* ω is a non-negative locally finite Borel measure.

The one weight inequality for the Hilbert transform is then

$$\|Hf\|_{L^2(\omega)} \lesssim \|f\|_{L^2(\omega)},$$

and was shown by Hunt, Muckenhoupt and Wheeden in [HuMuWh] to be equivalent to finiteness of the remarkable A_2 condition of Muckenhoupt: namely that $d\omega = w(x)dx$ is absolutely continuous with respect to Lebesgue measure and

$$\sup_I \frac{1}{|I|} \int_I w(x)dx \cdot \frac{1}{|I|} \int_I \frac{1}{w(x)}dx < \infty,$$

which says that the Cauchy-Schwarz inequality

$$1 = \left(\frac{1}{|I|} \int_I \sqrt{w(x) \frac{1}{w(x)}} dx \right)^2 \leq \frac{1}{|I|} \int_I w(x)dx \cdot \frac{1}{|I|} \int_I \frac{1}{w(x)}dx$$

can be reversed up to a constant uniformly over intervals I .

For two weights ω, σ , we consider the two weight norm inequality

$$(1.2) \quad \|H(f\sigma)\|_{L^2(\omega)} \leq \mathfrak{N} \|f\|_{L^2(\sigma)}. \quad (\mathfrak{N})$$

Note that the special case where ω is absolutely continuous with density w , and both w and $\frac{1}{w}$ are locally integrable, then the case $\sigma = \frac{1}{w}dx$ reduces to the one weight inequality above. A simple necessary condition for (1.2) to hold is the analogous two weight A_2 condition

$$\sup_I \frac{1}{|I|} \int_I d\omega(x) \cdot \frac{1}{|I|} \int_I d\sigma(x) < \infty,$$

taken uniformly over intervals I , where of course the weights need no longer be absolutely continuous. As it turns out though, this two weight A_2 condition is no longer sufficient for the norm inequality (1.2), and F. Nazarov has shown that even the following necessary \mathcal{A}_2 condition ‘on steroids’ of Nazarov, Treil and Volberg is not sufficient:

$$(1.3) \quad \sup_I P(I, \omega) \cdot P(I, \sigma) = \mathcal{A}_2 < \infty, \quad (\mathcal{A}_2)$$

where for an interval I and measure ω , the Poisson integral $P(I, \omega)$ at I is given by

$$(1.4) \quad P(I, \omega) \equiv \int_{\mathbb{R}} \frac{|I|}{(|I| + \text{dist}(x, I))^2} d\omega(x).$$

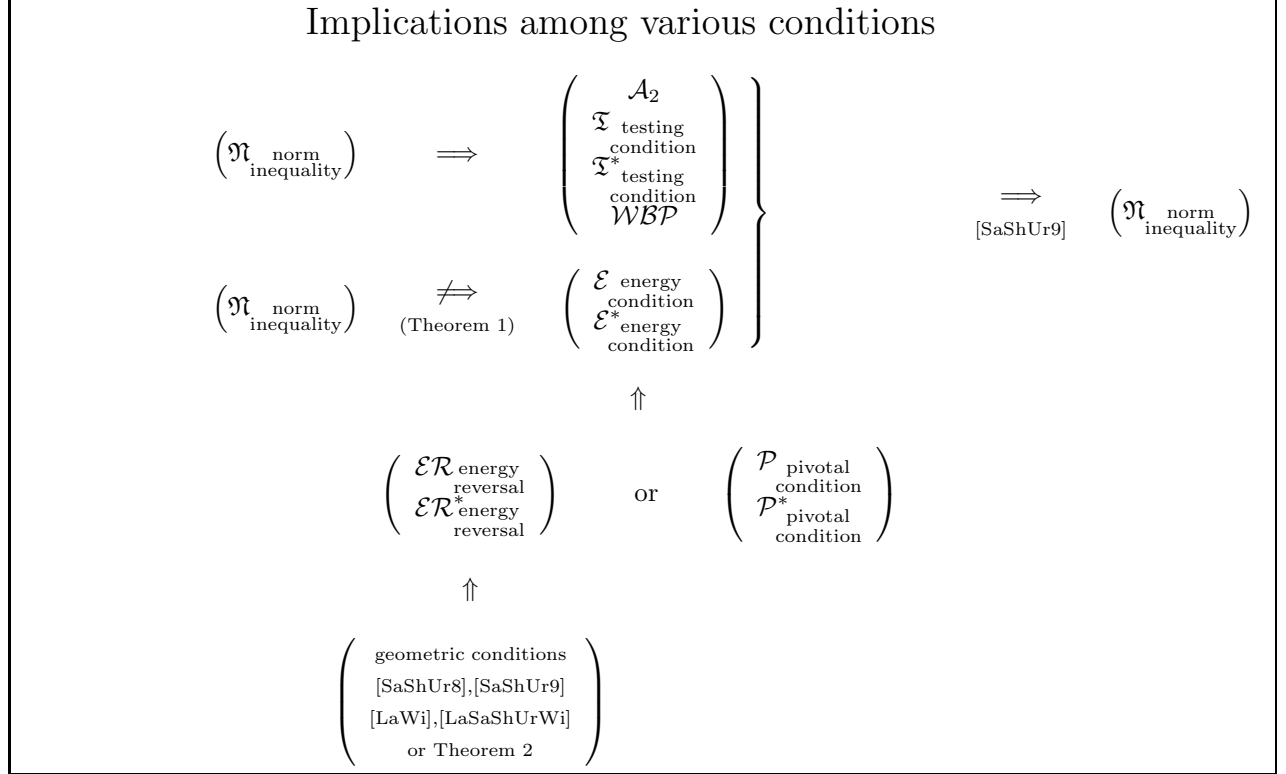
See e.g. Theorem 2.1 in [NiTr].

We require in addition that the following testing conditions, also necessary for the two weight inequality (1.2),

$$(1.5) \quad \int_I |H(\mathbf{1}_I \sigma)|^2 \omega(dx) \leq \mathfrak{T}^2 |I|_{\sigma}, \quad (\mathfrak{T})$$

$$(1.6) \quad \int_I |H(\mathbf{1}_I \omega)|^2 \sigma(dx) \leq (\mathfrak{T}^*)^2 |I|_{\omega}, \quad (\mathfrak{T}^*)$$

hold uniformly over intervals I . Here, we are letting \mathfrak{T} and \mathfrak{T}^* denote the smallest constants for which these inequalities are true uniformly over all intervals I , and we write $\sigma(I) \equiv \int_I \sigma(dx) \equiv |I|_\sigma$. The ‘NTV conjecture’ of Nazarov, Treil and Volberg was that (1.3) and the testing conditions (1.5,1.6) are also sufficient for the norm inequality (1.2). The following diagram illustrates the main connections between the conditions introduced here (for the definition of *energy reversal* see e.g. [SaShUr4]), but with arbitrary singular integrals in place of the Hilbert transform.



The approach of Nazarov, Treil and Volberg to prove (1.2) in [NTV4] involves the assumption of two additional side conditions, the *Pivotal Conditions* given by

$$(1.7) \quad \sum_{r=1}^{\infty} |I_r|_\omega \mathbf{P}(I_r, \mathbf{1}_{I_0} \sigma)^2 \leq \mathcal{P}^2 |I_0|_\sigma, \quad (\mathcal{P})$$

and its dual in which the measures σ and ω are interchanged, and where the inequality is required to hold for all intervals I_0 , and decompositions $\{I_r : r \geq 1\}$ of I_0 into disjoint intervals $I_r \subsetneq I_0$. Here \mathcal{P} and \mathcal{P}^* denote the best constants in (1.7) and its dual respectively.

In the approach initiated in [LaSaUr2], the Pivotal Condition (1.7) was replaced by certain weaker conditions of *energy type*.

Definition 1. For a weight ω , and interval I , we set

$$\mathbf{E}(I, \omega) \equiv \left[\mathbb{E}_I^{\omega(dx)} \left[\mathbb{E}_I^{\omega(dx')} \frac{(x - x')^2}{|I|} \right]^2 \right]^{1/2}.$$

It is important to note that $\mathbf{E}(I, \omega) \leq 1$, and can be quite small, if ω is highly concentrated inside the interval I ; in particular if $\omega \mathbf{1}_I$ is a point mass, then $\mathbf{E}(I, \omega) = 0$. Note also that $\omega(I) |I|^2 \mathbf{E}(I, \omega)^2$ is the variance of the variable x , and that we have the identity

$$\mathbf{E}(I, \omega)^2 = \frac{1}{2} \mathbb{E}_I^{\omega(dx)} \mathbb{E}_I^{\omega(dx')} \frac{(x - x')^2}{|I|^2}.$$

The following *Energy Condition* and its dual condition were shown in [LaSaUr2] to be necessary for the two weight norm inequality for the Hilbert transform:

$$(1.8) \quad \sum_{r \geq 1} |I_r|_{\omega} \mathbf{E}(I_r, \omega)^2 \mathbf{P}(I_r, \sigma \mathbf{1}_{I_0})^2 \leq \mathcal{E}^2 |I_0|_{\sigma}, \quad (\mathcal{E})$$

where the sum is taken over all decompositions $I_0 = \bigcup_{r=1}^{\infty} I_r$ of the interval I_0 into pairwise disjoint intervals $\{I_r\}_{r \geq 1}$. Here \mathcal{E} and \mathcal{E}^* denote the best constants in (1.8) and its dual respectively. As $\mathbf{E}(I, \omega) \leq 1$, the Energy Condition is weaker than the Pivotal Condition.

In [LaSaUr2] it was shown that the pivotal condition is **not** necessary for the norm inequality for the Hilbert transform. Later, in the two part paper [LaSaShUr3]-[Lac] by the authors and Lacey, the NTV conjecture was proved, namely that the two weight inequality for the Hilbert transform holds if and only if the \mathcal{A}_2 and testing conditions hold - a restriction that the measures have no common point masses was subsequently removed by Hytönen [Hyt2]. A key role was played in these results by the energy conditions, and in fact the energy conditions have continued to play a crucial role in higher dimensions.

For example, they arise as side conditions for the $T1$ theorem in our papers [SaShUr6] and [SaShUr7], where we raised the question of whether or not the energy conditions are necessary for *any* elliptic operator in higher dimensions. They also play a critical role in the side conditions of uniformly full dimension of Lacey and Wick [LaWi] since those conditions, like the doubling conditions in [NTV1], and the Energy Hypothesis in [LaSaUr2], imply the energy conditions. The energy conditions were also shown to be necessary, and used crucially, in the cases of the Hilbert transform [LaSaShUr3]-[Lac] already mentioned, in the Cauchy operator with one measure on a line or circle [LaSaShUrWi], and in the generalization to compact $C^{1,\delta}$ curves in higher dimension [SaShUr8].

In summary, the necessity of the energy condition was a crucial element in each of these proofs of the $T1$ theorem for two weights, and in fact in all proofs of a two weight $T1$ theorem to date. However, in every one of these cases, the energy conditions were derived from the very strong property of *energy reversal* for an operator. This property is tied in an essential way to the ellipticity of the gradient of the kernel of the operator, and to either the ‘fullness’ of doubling or uniformly full dimension measures, or to the ‘one-dimensional nature’ of one of the measures. Subsequently, Lacey observed the failure of energy reversal for the Cauchy operator in the plane, and shortly thereafter in [SaShUr4], we established that energy reversal fails spectacularly for classical fractional singular integrals in higher dimension. However, this left open the crucial question of whether or not the energy conditions themselves were necessary for (1.2) to hold. Prior to this paper, *no counterexample to the necessity of the crucial **energy conditions** has been found for any elliptic operator in any dimension.*

The main purpose of this paper then is to give such a counterexample.

Theorem 1. *There exists a weight pair $(\hat{\sigma}, \hat{\omega})$ and an elliptic singular integral H_b on the real line \mathbb{R} such that H_b satisfies the two weight norm inequality*

$$\|H_b(f\hat{\sigma})\|_{L^2(\hat{\omega})} \lesssim \|f\|_{L^2(\hat{\sigma})},$$

*yet the weight pair $(\hat{\sigma}, \hat{\omega})$ **fails** to satisfy the backward Energy Condition*

$$(1.9) \quad \sum_{r \geq 1} |I_r|_{\hat{\sigma}} \mathbf{E}(I_r, \hat{\sigma})^2 \mathbf{P}(I_r, \hat{\omega} \mathbf{1}_{I_0})^2 \leq (\mathcal{E}^*)^2 |I_0|_{\hat{\omega}}, \quad (\mathcal{E}^*)$$

for any $\mathcal{E}^ < \infty$, where the sum is taken over all decompositions $I_0 = \bigcup_{r=1}^{\infty} I_r$ of the interval I_0 into pairwise disjoint intervals $\{I_r\}_{r \geq 1}$.*

To prove this theorem, we build on the example from [LaSaUr2] that was used to show the failure of necessity of the pivotal condition for the Hilbert transform. However, there is a strong connection between testing conditions and their corresponding energy conditions, and we must work hard to disengage this connection while simultaneously retaining the norm inequality. All of this requires substantial modification of the example in [LaSaUr2], and involves delicate symmetries of redistributed Cantor measures. Moreover, it is essential to perturb the Hilbert transform so that it is no longer gradient elliptic. We now describe our strategy in more detail:

- As in [LaSaUr2], we start with one of the usual Cantor measures ω on $[0, 1]$, and a measure $\dot{\sigma}$ that consists of an infinite number of point masses centered in the gaps of the Cantor measure ω , and that is chosen so that the A_2 condition holds for the pair $(\dot{\sigma}, \omega)$.

- Now we replace the point masses in $\dot{\sigma}$ with averages over small intervals L to get a measure σ so that the A_2 condition holds for (σ, ω) , but the backward energy condition **fails** since it becomes the backward pivotal condition, which was essentially shown to fail in [LaSaUr2]. This depends crucially on the fact that the energy of a point mass vanishes, but not that of an average over an interval.
- However, the backward testing condition for the pair (σ, ω) then fails due to the strong connection between testing and energy conditions. In order to obtain the backward testing condition for the pair (σ, ω) , it would suffice to have our singular integral H_b applied to ω **vanish** on the intervals L , and in particular this requires the kernel K_b of our singular integral to have appropriately located flat spots.
- In order to achieve $H_b \omega = 0$ on the intervals L , it is thus necessary to inductively redistribute the Cantor measure ω into a new measure $\hat{\omega}$.
- To preserve the A_2 condition we must reweight the interval masses in σ to get a new measure $\hat{\sigma}$. It turns out that the backward testing condition continues to hold for the new pair $(\sigma, \hat{\omega})$ and that the backward energy condition continues to fail for the new pair $(\sigma, \hat{\omega})$.
- However, the forward testing condition is now in doubt because the argument in [LaSaUr2] for the analogous inequality made strong use of the self-similarity of both measures involved. The redistributed measures $\hat{\omega}$ and $\hat{\sigma}$ seem at first sight to have lost all trace of self-similarity. Surprisingly, there are hidden symmetries in the construction of $\hat{\omega}$ that lead to suitable replication formulas for both $\hat{\omega}$ and $\hat{\sigma}$, and then a delicate argument shows that the forward testing condition does indeed hold for the pair $(\hat{\sigma}, \hat{\omega})$. This step represents the main challenge overcome in this paper.
- Finally, we wish to prove that the norm inequality holds for the weight pair $(\hat{\sigma}, \hat{\omega})$, and the only known methods for this to date involve having both energy conditions for the weight pair under consideration. Since the backward energy condition fails for this weight pair, we must resort to a trick that exploits the flat spots of our kernel. The first half of the trick is to notice that everything we have done for the weight pair $(\hat{\sigma}, \hat{\omega})$ can also be done for the weight pair $(\hat{\hat{\sigma}}, \hat{\omega})$ in which $\hat{\hat{\sigma}}$ is the corresponding reweighting of the original measure $\dot{\sigma}$, but with one exception: the backward energy condition now **holds** because $\hat{\hat{\sigma}}$ consists of point masses instead of intervals, and because the energy of individual point masses vanishes. The second half of the trick is then to observe that the dual norm inequalities for the weight pairs $(\hat{\sigma}, \hat{\omega})$ and $(\hat{\hat{\sigma}}, \hat{\omega})$ are **equivalent** due to the flat spots in the kernel! Thus the norm inequality for H_b holds for the weight pair $(\hat{\hat{\sigma}}, \hat{\omega})$, but the backward energy condition fails for $(\hat{\sigma}, \hat{\omega})$.

As mentioned earlier, provided we narrowly avoid the operator H_b constructed in the proof of Theorem 1, the $T1$ theorem will hold.

Theorem 2. *Suppose that the kernel K of a standard singular integral T on the real line is gradient elliptic, i.e. $\left| \frac{\partial}{\partial x} K(x, y) \right|, \left| \frac{\partial}{\partial y} K(x, y) \right| \geq \frac{c}{(x-y)^2}$. Then $T : L^2(\sigma) \rightarrow L^2(\omega)$ if and only if the Muckenhoupt/NTV A_2 condition (1.3) holds, and both testing conditions hold,*

$$(1.10) \quad \int_I |T(\mathbf{1}_I \sigma)|^2 \omega(dx) \leq \mathfrak{T}^2 |I|_\sigma ,$$

$$(1.11) \quad \int_I |T^*(\mathbf{1}_I \omega)|^2 \sigma(dx) \leq (\mathfrak{T}^*)^2 |I|_\omega .$$

Theorems 1 and 2 point to a new phenomenon that is needed in order to obtain the energy conditions, namely the need for gradient ellipticity of the kernels of the singular integrals. This paper takes a first step toward the pursuit of $T1$ theorems via energy conditions.

We end this introduction by pointing to some applications of the two weight $T1$ theorem in operator theory, such as in [LaSaShUrWi], where embedding measures are characterized for model spaces K_θ , where θ is an inner function on the disk, and where norms of composition operators are characterized that map K_θ into Hardy and Bergman spaces. A $T1$ theorem could also have implications for a number of problems that are higher dimensional analogues of those connected to the Hilbert transform¹,

¹We thank Professor N. Nikolski for providing information on many of these topics.

- (1) when a rank one perturbation of a unitary operator is similar to a unitary operator (see e.g. [Vol] and [NiTr]): this could extend to an analogous question for a rank one perturbation of a normal operator T and lead to a two weight inequality for the Cauchy transform with one measure being the spectral measure of T ,
- (2) when a product of two densely defined Toeplitz operators $T_a T_b$ is a bounded operator, which is equivalent to the Birkhoff-Wiener-Hopf factorization for a given function $c = ab$; the same questions for the Bergman space could lead to a two weight problem for the Beurling transform,
- (3) questions regarding subspaces of the Hardy space invariant under the inverse shift operator (see e.g. [Vol] and [NaVo]),
- (4) questions concerning orthogonal polynomials (see e.g. [VoYu], [PeVoYu] and [PeVoYu1]),

and also to a variety of questions in quasiconformal theory (due to the relevance of the Beurling transform in that context) such as,

- (1) the conjecture of Iwaniec and Martin, at the level of Hausdorff dimension distortion (see [IwMa]) and, at the level of Hausdorff measures, higher dimensional analogues of the Astala conjecture (see e.g. [LaSaUr]), for which proof an (essentially) two weight inequality for the Beurling transform was crucial, and in general, similar questions pertaining to the higher dimensional analogues of the Beurling transform,
- (2) the problem of characterizing which Beltrami coefficients give rise to biLipschitz maps (see e.g. [AsGo] where a two weight theorem is proved for very specific weights),
- (3) and to the well known problem of the connectivity of the manifold of planar chord-arc curves (see e.g. [AsGo] and [AsZi]).

The proof of Theorem 1 will be given in the next four sections, and the proof of Theorem 2 will be given in the final section.

2. FAMILIES OF CANTOR MEASURES

Let $N \geq 2$. We construct a standard variant $\mathbf{E}^{(N)}$ of the middle thirds Cantor set \mathbf{E} , namely we remove the middle ‘ $\frac{N-2}{N}$ ’ at each generation, and we then construct an associated weight pair $(\hat{\sigma}, \hat{\omega})$, where $\hat{\omega}$ is a ‘redistribution’ of the Cantor measure $\omega^{(N)}$ associated with $\mathbf{E}^{(N)}$, and $\hat{\sigma}$ is mutually singular with respect to $\hat{\omega}$. We will later show that for N sufficiently large, there is an elliptic perturbation H_b of the Hilbert transform H , such that H_b is bounded from $L^2(\hat{\sigma})$ to $L^2(\hat{\omega})$, but the backward Energy Condition **fails** for the weight pair $(\hat{\sigma}, \hat{\omega})$.

2.1. The Cantor construction. We modify the construction of the middle-third Cantor set \mathbf{E} and Cantor measure ω on the closed unit interval $I_1^0 = [0, 1]$. Fix a real number

$$N > 2,$$

and at each stage, remove the central interval of proportion

$$\alpha \equiv \frac{N-2}{N}$$

(the usual Cantor construction is the case $N = 3$ and $\alpha = \frac{1}{3}$). At the k^{th} generation in the new construction, there is a collection $\{I_j^k\}_{j=1}^{2^k}$ of 2^k pairwise disjoint closed intervals of length $|I_j^k| = \frac{1}{N^k}$. With $K_k = \bigcup_{j=1}^{2^k} I_j^k$, the N -Cantor set is defined by

$$\mathbf{E}^{(N)} = \bigcap_{k=1}^{\infty} K_k = \bigcap_{k=1}^{\infty} \left(\bigcup_{j=1}^{2^k} I_j^k \right).$$

The N -Cantor measure $\omega^{(N)}$ is the unique probability measure supported in $\mathbf{E}^{(N)}$ with the property that it is equidistributed among the intervals $\{I_j^k\}_{j=1}^{2^k}$ at each scale k , i.e.

$$\omega^{(N)}(I_j^k) = 2^{-k}, \quad k \geq 0, 1 \leq j \leq 2^k.$$

Now we fix N large for the moment and suppress the dependence on N in our notation, e.g. we denote $\omega^{(N)}$ by simply ω . We denote the removed open middle α^{th} of I_j^k by G_j^k .

3. THE FLATTENED HILBERT TRANSFORM

We will now take an integer $N \geq 3$ large and a positive number $\rho \in (0, 1)$ close to 1, and flatten the convolution kernel $K(x) = \frac{1}{x}$ of the Hilbert transform in sufficiently small neighbourhoods of the points $\{\pm N^k\}_{k \in \mathbb{Z}}$ to obtain a flattened Hilbert transform H_b . Let $z_j^k \in G_j^k$ be the center of the interval G_j^k , which is also the center of the interval I_j^k . Then we will redistribute the measure ω constructed above into a new measure $\hat{\omega}$ supported on $E^{(N)}$ with the property that $H_b \hat{\omega}(z_j^k) = 0$ for all (k, j) . Then we define the weights \hat{s}_j^k so that the measure

$$\hat{\sigma} = \sum_{k,j} \hat{s}_j^k \delta_{z_j^k},$$

satisfies a certain ‘local’ \mathcal{A}_2 condition with respect to $\hat{\omega}$, and define

$$\hat{\sigma} \equiv \sum_{k,j} \hat{s}_j^k \frac{1}{|L_j^k|} \mathbf{1}_{L_j^k},$$

where L_j^k is a small interval centered at z_j^k .

We establish below that the weight pair $(\hat{\sigma}, \hat{\omega})$ satisfies the \mathcal{A}_2 condition, both the forward and backward testing conditions with respect to H_b , and the forward and backward energy conditions. Then we conclude from our theorem in [SaShUr7] (note that $\hat{\omega}$ and $\hat{\sigma}$ have no common point masses) that $(\hat{\sigma}, \hat{\omega})$ satisfies the norm inequality with respect to H_b , and from this we then deduce the norm inequality for the weight pair $(\hat{\sigma}, \hat{\omega})$ with respect to H_b . Finally we show that the weight pair $(\hat{\sigma}, \hat{\omega})$ **fails** to satisfy the backward energy condition.

3.1. The flattened Hilbert kernel. We define $K_b : \left[\frac{1}{\sqrt{N}}, \sqrt{N}\right] \rightarrow \left[\frac{1}{\sqrt{N}}, \sqrt{N}\right]$ to be smooth and satisfy

$$K_b(x) = \begin{cases} \frac{1}{x} & \text{if } \frac{1}{\sqrt{N}} \leq x \leq \frac{1}{\sqrt{\rho N}} \\ 1 & \text{if } \frac{1}{\sqrt{\rho^2 N}} \leq x \leq \sqrt{\rho^2 N} \\ \frac{1}{x} & \text{if } \sqrt{\rho N} \leq x \leq \sqrt{N} \end{cases}.$$

Then for $k \in \mathbb{Z}$, we extend the definition of K_b to the intervals $N^k \left[\frac{1}{\sqrt{N}}, \sqrt{N}\right] = \left[N^{k-\frac{1}{2}}, N^{k+\frac{1}{2}}\right]$ by $K_b(x) = N^{-k} K_b(N^{-k}x)$ for $x \in \left[N^{k-\frac{1}{2}}, N^{k+\frac{1}{2}}\right]$, and finally extend the definition of K_b to the entire real line \mathbb{R} by requiring it to be an odd function on \mathbb{R} . The resulting kernel K_b is smooth on $\mathbb{R} \setminus \{0\}$ and satisfies the standard CZ estimates, and most importantly, for each k the kernel K_b is flat on the interval

$$F_k \equiv N^{-k} \left[\frac{1}{\sqrt{\rho^2 N}}, \sqrt{\rho^2 N} \right] = \left[\frac{1}{\rho} N^{-k-\frac{1}{2}}, \rho N^{-k+\frac{1}{2}} \right]$$

containing the point N^{-k} .

Equally important is the connection with the measure ω constructed with parameter N . If $x \in I_r^\ell$ and $y \in I_j^k$ with $I_r^\ell \cap I_j^k = \emptyset$, and if $m \in \mathbb{N}$ is the largest positive integer such that both x and y belong to some I_i^m for $1 \leq i \leq 2^m$, then

$$(3.1) \quad \left| \frac{1}{N^m} - |x - y| \right| \leq \frac{2}{N^{m+1}}.$$

Define $I_{r,\text{left}}^\ell$ and $I_{r,\text{right}}^\ell$ to be the closest intervals $I_j^{\ell+1}$ on each side of z_r^ℓ that belong to the next generation. We claim that if

$$x \in I_{i,\text{left}}^m \cup \left[z_i^m - \frac{1}{N^{m+1}}, z_i^m + \frac{1}{N^{m+1}} \right], \quad y \in I_{i,\text{right}}^m,$$

then

$$(3.2) \quad |x - y| \in \left[\frac{1}{N^m} - \frac{2}{N^{m+1}}, \frac{1}{N^m} \right] \cup \left[\frac{1}{2N^m} - \frac{2}{N^{m+1}}, \frac{1}{2N^m} + \frac{1}{N^{m+1}} \right] \subset F_m,$$

provided

$$\frac{1}{\rho} N^{-m-\frac{1}{2}} \leq \frac{1}{2N^m} - \frac{2}{N^{m+1}} \text{ and } \frac{1}{N^m} \leq \rho N^{-m+\frac{1}{2}},$$

equivalently

$$\frac{1}{\rho} \frac{1}{\sqrt{N}} \leq \frac{1}{2} - \frac{2}{N} \text{ and } 1 \leq \rho\sqrt{N},$$

which holds for example if $N \geq 16$ and $\rho \geq \frac{2}{3}$.

4. SELF-SIMILAR MEASURES

4.1. The redistributed measure $\widehat{\omega}$. We now construct, by induction on the level ℓ , new measures ω_ℓ by adjusting at each stage the relative weighting of the measure ω_ℓ on these two intervals $I_{r,\text{left}}^\ell$ and $I_{r,\text{right}}^\ell$. We begin with $\omega_1 \equiv \omega$, and having defined ω_ℓ inductively we define $\omega_{\ell+1}$ as follows. Fix the measure ω_ℓ that was constructed inductively, and note that it is supported on the Cantor set $\mathbf{E}^{(N)}$ constructed in the previous section. At the point $y = z_r^\ell$, and with $x_{r,\text{left}}^\ell$ and $x_{r,\text{right}}^\ell$ denoting any points in $I_{r,\text{left}}^\ell$ and $I_{r,\text{right}}^\ell$ respectively, we have from (3.2) that

$$\begin{aligned} (4.1) \quad H_b \omega_\ell(z_r^\ell) &= \int_{(G_r^\ell)^c} K_b(x - z_r^\ell) d\omega_\ell(x) = \sum_i \int_{I_i^{\ell+1}} K_b(x - z_r^\ell) d\omega_\ell(x) \\ &= \int_{I_{r,\text{left}}^\ell} K_b(x - z_r^\ell) d\omega_\ell(x) + \int_{I_{r,\text{right}}^\ell} K_b(x - z_r^\ell) d\omega_\ell(x) + \sum_{i: I_i^{\ell+1} \not\subseteq \{I_{r,\text{left}}^\ell, I_{r,\text{right}}^\ell\}} \int_{I_i^{\ell+1}} K_b(x - z_r^\ell) d\omega_\ell(x) \\ &= K_b(x_{r,\text{left}}^\ell - z_r^\ell) |I_{r,\text{left}}^\ell|_{\omega_\ell} + K_b(x_{r,\text{right}}^\ell - z_r^\ell) |I_{r,\text{right}}^\ell|_{\omega_\ell} + \sum_{i: I_i^{\ell+1} \not\subseteq \{I_{r,\text{left}}^\ell, I_{r,\text{right}}^\ell\}} \int_{I_i^{\ell+1}} K_b(x - z_r^\ell) d\omega_\ell(x) \\ &= K_b(x_{r,\text{left}}^\ell - z_r^\ell) \left(|I_{r,\text{left}}^\ell|_{\omega_\ell} - |I_{r,\text{right}}^\ell|_{\omega_\ell} \right) + \sum_{i: I_i^{\ell+1} \not\subseteq \{I_{r,\text{left}}^\ell, I_{r,\text{right}}^\ell\}} \int_{I_i^{\ell+1}} K_b(x - z_r^\ell) d\omega_\ell(x), \end{aligned}$$

where we recall that $I_{r,\text{left}}^\ell$ and $I_{r,\text{right}}^\ell$ are the closest next generation intervals $I_j^{\ell+1}$ on each side of z_r^ℓ . Now we define the relative weighting of the measure $\omega_{\ell+1}$ on these two intervals $I_{r,\text{left}}^\ell$ and $I_{r,\text{right}}^\ell$ by taking $\omega_{\ell+1}$ to satisfy both

$$\begin{aligned} (4.2) \quad |I_r^\ell|_{\omega_{\ell+1}} &= |I_{r,\text{left}}^\ell|_{\omega_{\ell+1}} + |I_{r,\text{right}}^\ell|_{\omega_{\ell+1}} = |I_{r,\text{left}}^\ell|_{\omega_\ell} + |I_{r,\text{right}}^\ell|_{\omega_\ell} = |I_r^\ell|_{\omega_\ell}, \\ K_b(x_{r,\text{left}}^\ell - z_r^\ell) \left(|I_{r,\text{left}}^\ell|_{\omega_{\ell+1}} - |I_{r,\text{right}}^\ell|_{\omega_{\ell+1}} \right) &= - \sum_{i: I_i^{\ell+1} \not\subseteq \{I_{r,\text{left}}^\ell, I_{r,\text{right}}^\ell\}} \int_{I_i^{\ell+1}} K_b(x - z_r^\ell) d\omega_\ell(x). \end{aligned}$$

Thus we have

$$\begin{aligned} (4.3) \quad |I_{r,\text{left}}^\ell|_{\omega_{\ell+1}} &= \frac{1}{2} \left\{ |I_r^\ell|_{\omega_\ell} - \sum_{i: I_i^{\ell+1} \not\subseteq \{I_{r,\text{left}}^\ell, I_{r,\text{right}}^\ell\}} \int_{I_i^{\ell+1}} \frac{K_b(x - z_r^\ell)}{K_b(x_{r,\text{left}}^\ell - z_r^\ell)} d\omega_\ell(x) \right\}, \\ |I_{r,\text{right}}^\ell|_{\omega_{\ell+1}} &= \frac{1}{2} \left\{ |I_r^\ell|_{\omega_\ell} + \sum_{i: I_i^{\ell+1} \not\subseteq \{I_{r,\text{left}}^\ell, I_{r,\text{right}}^\ell\}} \int_{I_i^{\ell+1}} \frac{K_b(x - z_r^\ell)}{K_b(x_{r,\text{left}}^\ell - z_r^\ell)} d\omega_\ell(x) \right\}. \end{aligned}$$

At this point we note that in analogy with (4.1) we have

$$H_b \omega_{\ell+1}(z_r^\ell) = K_b(x_{r,\text{left}}^\ell - z_r^\ell) \left(|I_{r,\text{left}}^\ell|_{\omega_{\ell+1}} - |I_{r,\text{right}}^\ell|_{\omega_{\ell+1}} \right) + \sum_{i: I_i^{\ell+1} \not\subseteq \{I_{r,\text{left}}^\ell, I_{r,\text{right}}^\ell\}} \int_{I_i^{\ell+1}} K_b(x - z_r^\ell) d\omega_{\ell+1}(x),$$

and then the second line in (4.2) gives

$$H_b \omega_{\ell+1}(z_r^\ell) = - \sum_{i: I_i^{\ell+1} \not\subseteq \{I_{r,\text{left}}^\ell, I_{r,\text{right}}^\ell\}} \int_{I_i^{\ell+1}} K_b(x - z_r^\ell) d\omega_\ell(x) + \sum_{i: I_i^{\ell+1} \not\subseteq \{I_{r,\text{left}}^\ell, I_{r,\text{right}}^\ell\}} \int_{I_i^{\ell+1}} K_b(x - z_r^\ell) d\omega_{\ell+1}(x).$$

However, the intervals $I_i^{\ell+1} \not\subseteq \{I_{r,\text{left}}^\ell, I_{r,\text{right}}^\ell\}$ can be grouped into pairs $I_k^{\ell+1}, I_{k'}^{\ell+1}$ that are the children of the same interval I_s^ℓ with $s \neq r$, and by (3.2) we have that $K_b(x - z_r^\ell)$ is *constant* on the parent I_s^ℓ of $I_k^{\ell+1}$

and $I_{k'}^{\ell+1}$. Thus by the first line in (4.2), we conclude that

$$\begin{aligned} & - \sum_{i=k,k'} \int_{I_i^{\ell+1}} K_b(x - \dot{z}_r^\ell) d\omega_\ell(x) + \sum_{i=k,k'} \int_{I_i^{\ell+1}} K_b(x - \dot{z}_r^\ell) d\omega_{\ell+1}(x) \\ & = K_b(\dot{z}_s^\ell - \dot{z}_r^\ell) \left(-|I_k^{\ell+1}|_{\omega_\ell} - |I_{k'}^{\ell+1}|_{\omega_\ell} + |I_k^{\ell+1}|_{\omega_{\ell+1}} + |I_{k'}^{\ell+1}|_{\omega_{\ell+1}} \right) = 0. \end{aligned}$$

Then summing over all $s \neq r$ we obtain the key consequence

$$(4.4) \quad H_b \omega_{\ell+1}(\dot{z}_r^\ell) = 0.$$

We will also see as an immediate consequence of Lemma 1 below that the ratios $\frac{|I_{r,\text{left}}^\ell|_{\omega_{\ell+1}}}{|I_{r,\text{left}}^\ell|_{\omega_\ell}}$ and $\frac{|I_{r,\text{right}}^\ell|_{\omega_{\ell+1}}}{|I_{r,\text{right}}^\ell|_{\omega_\ell}}$ are one of the two values $1 \pm \frac{1}{N}$.

Now we take the limit as $\ell \rightarrow \infty$ of the measures ω_ℓ to get the measure

$$\widehat{\omega} \equiv \lim_{\ell \rightarrow \infty} \omega_\ell$$

that is supported on the Cantor set $E^{(N)}$. Finally, we define the interval L_j^k by

$$(4.5) \quad L_j^k \equiv [\dot{z}_j^k - N^{-k-1}, \dot{z}_j^k + N^{-k-1}],$$

and for $N \geq 16$ and $\rho \geq \frac{2}{3}$ as above, we obtain from (4.4) and (3.2) the following crucial estimate for $y \in L_j^k$:

$$(4.6) \quad H_b \widehat{\omega}(y) = 0, \quad \text{for } y \in L_j^k.$$

From (4.3) we have that $|I_{r,\text{left}}^\ell|_{\widehat{\omega}} = \frac{1+\eta}{2} |I_r^\ell|_{\widehat{\omega}}$ and $|I_{r,\text{right}}^\ell|_{\widehat{\omega}} = \frac{1-\eta}{2} |I_r^\ell|_{\widehat{\omega}}$ for some positive number $\eta = \eta(I_r^\ell)$ depending on the pair (ℓ, r) . We end this subsection by showing that $\eta(I_r^\ell)$ is the constant $\frac{1}{N}$. For this it will be convenient to introduce some tree notation.

4.1.1. The dyadic tree. Let $(\mathcal{T}, \succ, \mathfrak{r}, \pi, \mathfrak{C})$ denote the dyadic tree with relation \succ , root \mathfrak{r} , parent $\pi : \mathcal{T} \setminus \{\mathfrak{r}\} \rightarrow \mathcal{T}$ and children $\mathfrak{C} : \mathcal{T} \rightarrow \mathcal{T} \times \mathcal{T}$ satisfying the following properties:

- (1) \mathcal{T} is countable,
- (2) \succ is a partial order on \mathcal{T} , i.e. \succ is reflexive ($\alpha \succ \alpha$), antisymmetric ($\alpha \succ \beta$ implies $\beta \not\succ \alpha$), and transitive ($\alpha \succ \beta$ and $\beta \succ \gamma$ implies $\alpha \succ \gamma$),
- (3) $\mathfrak{r} \succ \alpha$ for all $\alpha \in \mathcal{T}$,
- (4) $\pi \alpha$ is the unique minimal element of $\{\beta \in \mathcal{T} : \beta \succ \alpha\}$ for all $\alpha \in \mathcal{T}$,
- (5) $\mathfrak{C}(\alpha) = (\alpha_-, \alpha_+)$ where α_- and α_+ are the unique pair of maximal elements of $\{\beta \in \mathcal{T} : \beta \prec \alpha\}$.

Define the *distance* $d(\alpha, \beta)$ between two points α, β in \mathcal{T} to be the number of steps taken to reach β from α by travelling along the unique geodesic $\overrightarrow{\alpha\beta}$ connecting α to β . Then define the *depth* or *level* of α to be $d(\alpha) = d(\mathfrak{r}, \alpha)$. We refer to $\pi \alpha$ as the *parent* of α , to $S_\alpha \equiv \{\beta \in \mathcal{T} : \beta \prec \alpha\}$ as the *successor set* of α , and to α_- and α_+ as the *left* and *right child* of α respectively. For convenience we often write $\pi^1 \alpha = \pi \alpha$, $\pi^2 \alpha = \pi \pi \alpha$, etc. and $(\alpha_\pm)_\pm = \alpha_{\pm\pm}$, $(\alpha_{\pm\pm})_\pm = \alpha_{\pm\pm\pm}$ etc. More generally, for $\epsilon \in \{+, -\}^m$, we denote by α_ϵ the point (called an order m -grandchild) $\alpha_{\pm\pm\ldots\pm}$ at depth m below α that is given by the choice of \pm as determined by the sequence ϵ . Finally, for any $\alpha \in \mathcal{T} \setminus \{\mathfrak{r}\}$, we define the sibling $\theta \alpha$ of α to be the other child of $\pi \alpha$. The relevant example of a dyadic tree here is the following dyadic tree defined in terms of our construction of intervals I_r^ℓ above, and with partial order defined in terms of set inclusion $I \prec J$ when $I \subset J$:

$$\mathcal{D} = \{I_r^\ell : (\ell, r) \in \mathbb{Z}_+ \times \mathbb{N}, 1 \leq r \leq 2^\ell\} \sim \{(\ell, r) \in \mathbb{Z}_+ \times \mathbb{N} : 1 \leq r \leq 2^\ell\}.$$

This tree has an additional structure that is derived from its representation as a collection of intervals on the line - namely the two children of I occur as a left child I_- and a right child I_+ on the line. Then $I_{\pm\pm}$ represent the four grandchildren, etc. We refer to trees with this additional left and right child structure as *embedded trees* since they can be drawn in the obvious way in the plane.

Lemma 1. *We have $\eta(I) = \frac{1}{N}$ for all $I \in \mathcal{D}$, and in addition,*

$$(4.7) \quad \begin{aligned} |I_{--}|_{\widehat{\omega}} &= \frac{1+\eta}{2} |I_-|_{\widehat{\omega}} \quad \text{and} \quad |I_{-+}|_{\widehat{\omega}} = \frac{1-\eta}{2} |I_-|_{\widehat{\omega}}, \\ |I_{+-}|_{\widehat{\omega}} &= \frac{1-\eta}{2} |I_+|_{\widehat{\omega}} \quad \text{and} \quad |I_{++}|_{\widehat{\omega}} = \frac{1+\eta}{2} |I_+|_{\widehat{\omega}}. \end{aligned}$$

Proof. Let c_I be the center of an interval I , and let \mathcal{D} be the tree of all the intervals I_r^ℓ under containment. For $I \in \mathcal{D}$ we denote by I_- and I_+ the left and right children of I in \mathcal{D} , and thus I_{--} , I_{-+} , I_{+-} and I_{++} are the four grandchildren of I in \mathcal{D} from left to right. We prove the following statement by induction on ℓ :

$$\eta(K) = \frac{1}{N} \text{ for } 0 < d(K) \leq \ell \text{ and } H_b \widehat{\omega}(c_I) = 0 \text{ for } d(I) = \ell, \quad K, I \in \mathcal{D}.$$

The case $\ell = 0$ holds trivially. Assume now that the induction statement holds for a fixed $\ell \geq 0$, and fix an interval $I = I_r^\ell \in \mathcal{T}$ at level ℓ in the tree, so that $I_- = I_{r,\text{left}}^\ell$ and $I_+ = I_{r,\text{right}}^\ell$. Then we have

$$0 = H_b \widehat{\omega}(c_I) = H_b(\mathbf{1}_{I^c} \widehat{\omega})(c_I) + H_b(\mathbf{1}_{I_-} \widehat{\omega})(c_I) + H_b(\mathbf{1}_{I_+} \widehat{\omega})(c_I).$$

Now, using the ‘flat spots’ in the construction of K_b , we obtain both $H_b(\mathbf{1}_{I^c} \widehat{\omega})(c_{I_+}) = H_b(\mathbf{1}_{I^c} \widehat{\omega})(c_I)$ and $H_b(\mathbf{1}_{I_-} \widehat{\omega})(c_{I_+}) = H_b(\mathbf{1}_{I_-} \widehat{\omega})(c_I)$. Using this in line 2, and the previous display in line 3, we obtain

$$\begin{aligned} H_b \widehat{\omega}(c_{I_+}) &= H_b(\mathbf{1}_{I^c} \widehat{\omega})(c_{I_+}) + H_b(\mathbf{1}_{I_-} \widehat{\omega})(c_{I_+}) + H_b(\mathbf{1}_{I_+} \widehat{\omega})(c_{I_+}) \\ &= H_b(\mathbf{1}_{I^c} \widehat{\omega})(c_I) + H_b(\mathbf{1}_{I_-} \widehat{\omega})(c_I) + H_b(\mathbf{1}_{I_+} \widehat{\omega})(c_{I_+}) \\ &= -H_b(\mathbf{1}_{I_+} \widehat{\omega})(c_I) + H_b(\mathbf{1}_{I_+} \widehat{\omega})(c_{I_+}) \\ &= -H_b(\mathbf{1}_{I_+} \widehat{\omega})(c_I) + H_b(\mathbf{1}_{I_{++}} \widehat{\omega})(c_{I_+}) + H_b(\mathbf{1}_{I_{++}} \widehat{\omega})(c_{I_+}) \\ &= -N^\ell |I_+|_{\widehat{\omega}} - N^{\ell+1} |I_{+-}|_{\widehat{\omega}} + N^{\ell+1} |I_{++}|_{\widehat{\omega}}, \end{aligned}$$

where in the last line we have used the ‘flat spots’ in the construction of K_b . Thus in order to achieve $H_b \widehat{\omega}(c_{I_+}) = 0$ we need

$$|I_{++}|_{\widehat{\omega}} - |I_{+-}|_{\widehat{\omega}} = \frac{1}{N} |I_+|_{\widehat{\omega}},$$

and combined with the requirement that $|I_{++}|_{\widehat{\omega}} + |I_{+-}|_{\widehat{\omega}} = |I_+|_{\widehat{\omega}}$, we obtain

$$|I_{++}|_{\widehat{\omega}} = \frac{1 + \frac{1}{N}}{2} |I_+|_{\widehat{\omega}} \text{ and } |I_{+-}|_{\widehat{\omega}} = \frac{1 - \frac{1}{N}}{2} |I_+|_{\widehat{\omega}},$$

which establishes the second line in (4.7) for I , and also that $\eta(I_{++}^\ell) = \eta(I_+) = \frac{1}{N}$ and $H_b \widehat{\omega}(c_{I_+}) = 0$.

Similarly, we have the first line in (4.7) for I and $\eta(I_{--}^\ell) = \frac{1}{N}$ and $H_b \widehat{\omega}(c_{I_-}) = 0$, which completes the proof of the inductive step. \square

4.2. The reweighted measures $\widehat{\sigma}$ and $\widehat{\sigma}$. Recall that we view the collection of intervals $\{I_j^k\}$ as an embedded tree and use the usual parent/child terminology for this tree. In Lemma 1 above we showed that the pattern of redistribution is given by

$$(4.8) \quad \begin{aligned} |I_{r,\text{left}}^\ell|_{\widehat{\omega}} &= \frac{1+\eta}{2} |I_r^\ell|_{\widehat{\omega}} \text{ and } |I_{r,\text{right}}^\ell|_{\widehat{\omega}} = \frac{1-\eta}{2} |I_r^\ell|_{\widehat{\omega}} \text{ if } I_r^\ell = I_{s,\text{left}}^{\ell-1}, \\ |I_{r,\text{left}}^\ell|_{\widehat{\omega}} &= \frac{1-\eta}{2} |I_r^\ell|_{\widehat{\omega}} \text{ and } |I_{r,\text{right}}^\ell|_{\widehat{\omega}} = \frac{1+\eta}{2} |I_r^\ell|_{\widehat{\omega}} \text{ if } I_r^\ell = I_{s,\text{right}}^{\ell-1}, \end{aligned}$$

where $I_s^{\ell-1} = \pi I_r^\ell$ is the parent of I_r^ℓ and where $\eta = \frac{1}{N}$. Thus we see that on the children of I_r^ℓ the measure $\widehat{\omega}$ is redistributed **away from the center of the parent of I_r^ℓ** , where the factors $\frac{1 \pm \eta}{2}$ can be remembered via the mnemonic 4-tuple

$$(4.9) \quad +, -, -, +$$

As a consequence of this redistribution, we see that the measures $|I_r^\ell|_{\widehat{\omega}}$ of the intervals I_r^ℓ at level ℓ are equal when $\ell = 1$ and thereafter take on the values

$$(4.10) \quad |I_r^\ell|_{\widehat{\omega}} = \frac{1}{2} \left(\frac{1+\eta}{2} \right)^H \left(\frac{1-\eta}{2} \right)^T = (1+\eta)^H (1-\eta)^T |I_r^\ell|_{\omega}, \quad H+T = \ell-1,$$

where $H = H(r)$ and $T = T(r)$ depend on r , and can be thought of as the number of heads and tails respectively in $\ell-1$ tosses of a fair coin.

We now define the weights \widehat{s}_j^k so that with

$$\widehat{\sigma} \equiv \sum_{k,j} \widehat{s}_j^k \delta_{z_j^k}$$

we have

$$(4.11) \quad \frac{|I_j^k|_{\hat{\omega}} \hat{s}_j^k}{N^{-2k}} = \frac{|I_j^k|_{\hat{\omega}} |G_j^k|_{\hat{\sigma}}}{|I_j^k|^2} = 1.$$

Then we replace the point mass $\delta_{z_j^k}$ at z_j^k in the definition of $\hat{\sigma}$ with the approximation $\frac{1}{|L_j^k|} \mathbf{1}_{L_j^k}$ and define the resulting reweighted measure $\hat{\sigma}$ by

$$\hat{\sigma} \equiv \sum_{k,j} \hat{s}_j^k \frac{1}{|L_j^k|} \mathbf{1}_{L_j^k}.$$

We now investigate properties of the measure pairs $(\hat{\sigma}, \hat{\omega})$ and $(\hat{\sigma}, \hat{\omega})$ relative to the flattened Hilbert transform H_b .

5. TESTING AND SIDE CONDITIONS

In this section we establish the Muckenhoupt/NTV \mathcal{A}_2 conditions for $(\hat{\sigma}, \hat{\omega})$, as well as the testing conditions for H_b relative to the weight pair $(\hat{\sigma}, \hat{\omega})$. Then we establish both the forward and backward energy conditions for the weight pair $(\hat{\sigma}, \hat{\omega})$, and use our $T1$ theorem in [SaShUr7] to conclude that the two weight norm inequality holds for the flattened Hilbert transform H_b relative to the weight pair $(\hat{\sigma}, \hat{\omega})$. Then using that $H_b(g\hat{\omega})$ is constant on the support of $\hat{\sigma}$ for all $g \in L^2(\hat{\omega})$, and that the support of $\hat{\sigma}$ contains the support of $\hat{\sigma}$, we establish that

$$\int |H_b(g\hat{\omega})|^2 d\hat{\sigma} = \int |H_b(g\hat{\omega})|^2 d\hat{\sigma}.$$

We thus conclude that the norm inequality also holds for H_b relative to the weight pair $(\hat{\sigma}, \hat{\omega})$.

Finally we will show that the backward energy condition fails for the weight pair $(\hat{\sigma}, \hat{\omega})$ since the support of $\hat{\sigma}$ consists of a countable union of *intervals* rather than *point masses* as is the case for the measure $\hat{\sigma}$. It is the fact that the energy of a measure on an interval is positive provided the measure has positive density on the interval, that accounts for the failure of the backward energy condition. And it is the fact that the flattened Hilbert transform of $\hat{\omega}$ *vanishes* on the support of $\hat{\sigma}$ that permits the backward testing condition to hold **despite** the failure of the backward energy condition. *This is the key reason for the failure of the energy condition to be necessary for the norm inequality of an elliptic singular integral.*

We can of course view this one-dimensional measure pair as living in the plane, and then the flattened Hilbert transform is the restriction to the x -axis of a suitable flattening R_{1b} of the Riesz transform R_1 . Thus the vector $\mathbf{R} = (R_{1b}, R_2)$ is a strongly elliptic singular integral that satisfies the norm inequality relative to $(\hat{\sigma}, \hat{\omega})$ but fails the backward energy condition, thus extending the failure of necessity of energy conditions to higher dimensions. One can easily extend this failure to hold for flattened fractional Riesz transforms $\mathbf{R}^{\alpha,n}$ in the Euclidean space \mathbb{R}^n as well.

5.1. The Muckenhoupt/NTV condition. We now verify the two-tailed Muckenhoupt/NTV condition \mathcal{A}_2 for the weight pair $(\hat{\sigma}, \hat{\omega})$. We fix an interval I_r^ℓ at level ℓ . Recall that $\eta = \frac{1}{N}$ by Lemma 1. Consider the two-tailed Muckenhoupt condition \mathcal{A}_2 . We write

$$P(Q, \mu) = \int_{\mathbb{R}^n} \frac{|Q|}{(|Q| + |y - c_Q|)^2} d\mu(y) = \int_{\mathbb{R}^n} \frac{1}{|Q|} \frac{1}{\left(1 + \frac{|y - c_Q|}{|Q|}\right)^2} d\mu(y),$$

where c_Q is the center of the interval Q , and we will use the pointwise estimates

$$(5.1) \quad \begin{aligned} \frac{1}{|Q|} \frac{1}{\left(1 + \frac{|y - c_Q|}{|Q|}\right)^2} &\approx \sum_{k=0}^{\infty} \frac{1}{N^k} \frac{1}{|N^k Q|} \mathbf{1}_{N^k Q}(y) \\ &\approx \frac{1}{|Q|} \mathbf{1}_Q(y) + \sum_{k=1}^{\infty} \frac{1}{N^k} \frac{1}{|N^k Q|} \mathbf{1}_{N^k Q \setminus N^{k-1} Q}(y), \end{aligned}$$

where the implied constants in \approx depend only on N . We compute both $P(I_r^\ell, \hat{\omega})$ and $P(I_r^\ell, \hat{\sigma})$, beginning with the simpler factor $P(I_r^\ell, \hat{\omega})$.

Using the second line in (5.1) we have

$$P(I_r^\ell, \hat{\omega}) \approx \frac{|I_r^\ell|_{\hat{\omega}}}{|I_r^\ell|} + \frac{1}{N^2} \frac{|\theta I_r^\ell|_{\hat{\omega}}}{|I_r^\ell|} + \frac{1}{N^4} \frac{|\theta \pi I_r^\ell|_{\hat{\omega}}}{|I_r^\ell|} + \frac{1}{N^6} \frac{|\theta \pi^{(2)} I_r^\ell|_{\hat{\omega}}}{|I_r^\ell|} + \dots$$

Our construction above gives the inequalities

$$(5.2) \quad \left| \theta \pi^{(k)} I_r^\ell \right|_{\hat{\omega}} \leq \frac{1+\eta}{1-\eta} \left| \pi^{(k)} I_r^\ell \right|_{\hat{\omega}} \quad \text{and} \quad \left| \pi^{(k+1)} I_r^\ell \right|_{\hat{\omega}} \leq \frac{2}{1-\eta} \left| \pi^{(k)} I_r^\ell \right|_{\hat{\omega}}, \quad k \geq 0,$$

and these inequalities show that for $N \geq 4$ we have

$$\begin{aligned} P(I_r^\ell, \hat{\omega}) &\lesssim \frac{|I_r^\ell|_{\hat{\omega}}}{|I_r^\ell|} \left\{ 1 + \frac{1}{N^2} \left(\frac{1+\eta}{1-\eta} \right) + \frac{1}{N^4} \left(\frac{1+\eta}{1-\eta} \right) \left(\frac{2}{1-\eta} \right) + \frac{1}{N^6} \left(\frac{1+\eta}{1-\eta} \right) \left(\frac{2}{1-\eta} \right)^2 + \dots \right\} \\ &= \frac{|I_r^\ell|_{\hat{\omega}}}{|I_r^\ell|} \left\{ 1 + \frac{1}{N^2} \left(\frac{1+\eta}{1-\eta} \right) \left[1 + \frac{1}{N^2} \left(\frac{2}{1-\eta} \right) + \frac{1}{N^4} \left(\frac{2}{1-\eta} \right)^2 + \dots \right] \right\} \\ &= \frac{|I_r^\ell|_{\hat{\omega}}}{|I_r^\ell|} \left\{ 1 + \frac{1}{N^2} \left(\frac{1+\eta}{1-\eta} \right) \frac{1}{1 - \frac{1}{N^2} \left(\frac{2}{1-\eta} \right)} \right\} = \frac{|I_r^\ell|_{\hat{\omega}}}{|I_r^\ell|} \left\{ 1 + \frac{1}{N^2} \frac{N+1}{N-1-\frac{2}{N}} \right\}, \end{aligned}$$

and hence that

$$P(I_r^\ell, \hat{\omega}) \approx \frac{|I_r^\ell|_{\hat{\omega}}}{|I_r^\ell|}.$$

Similarly, using the first line in (5.1), we have

$$(5.3) \quad P(I_r^\ell, \hat{\sigma}) \approx \frac{|I_r^\ell|_{\hat{\sigma}}}{|I_r^\ell|} + \frac{1}{N^2} \frac{|\pi I_r^\ell|_{\hat{\sigma}}}{|I_r^\ell|} + \frac{1}{N^4} \frac{|\pi^{(2)} I_r^\ell|_{\hat{\sigma}}}{|I_r^\ell|} + \frac{1}{N^6} \frac{|\pi^{(3)} I_r^\ell|_{\hat{\sigma}}}{|I_r^\ell|} + \dots$$

Now we show that $|I_r^\ell|_{\hat{\sigma}}$ is comparable to $|G_r^\ell|_{\hat{\sigma}} = \hat{s}_r^\ell$. For this it will be convenient to denote G_r^ℓ by $G(I_r^\ell)$ and \hat{s}_r^ℓ by $\hat{s}(I_r^\ell)$ in order to use the notations $G(\pi^{(k)} I_r^\ell)$ and $\hat{s}(\pi^{(k)} I_r^\ell)$.

Lemma 2. *For $N \geq 16$ we have*

$$|I_r^\ell|_{\hat{\sigma}} = \frac{N^2-1}{N^2-5} |G_r^\ell|_{\hat{\sigma}} \quad \text{for all } \ell \geq 1 \text{ and } 1 \leq r \leq 2^\ell.$$

Proof. Let $I = I_r^\ell \in \mathcal{D}$ with $\ell \geq 1$ and write $\hat{s}(I) = \hat{s}(I_r^\ell) \equiv \hat{s}_r^\ell$. From the definition $|I_r^\ell|_{\hat{\omega}} |G_r^\ell|_{\hat{\sigma}} = |I_r^\ell|^2$ of $\hat{s}_r^\ell = |G_r^\ell|_{\hat{\sigma}}$, we have that

$$\begin{aligned} |I|_{\hat{\sigma}} &= \hat{s}(I) + \{\hat{s}(I_-) + \hat{s}(I_+)\} + \{\hat{s}(I_{--}) + \hat{s}(I_{-+}) + \hat{s}(I_{+-}) + \hat{s}(I_{++})\} + \dots \\ &= \frac{|I|^2}{|I|_{\hat{\omega}}} + \left\{ \frac{|I_-|^2}{|I_-|_{\hat{\omega}}} + \frac{|I_+|^2}{|I_+|_{\hat{\omega}}} \right\} + \left\{ \frac{|I_{--}|^2}{|I_{--}|_{\hat{\omega}}} + \frac{|I_{-+}|^2}{|I_{-+}|_{\hat{\omega}}} + \frac{|I_{+-}|^2}{|I_{+-}|_{\hat{\omega}}} + \frac{|I_{++}|^2}{|I_{++}|_{\hat{\omega}}} \right\} + \dots \end{aligned}$$

and if we combine $|J_+|_{\hat{\omega}} = \frac{1+\eta}{2} |J|_{\hat{\omega}}$ and $|J_-|_{\hat{\omega}} = \frac{1-\eta}{2} |J|_{\hat{\omega}}$ with

$$\frac{|J|_{\hat{\omega}}}{|J_-|_{\hat{\omega}}} + \frac{|J|_{\hat{\omega}}}{|J_+|_{\hat{\omega}}} = \frac{2}{1+\eta} + \frac{2}{1-\eta} = \frac{4}{1-\eta^2} \quad \text{and} \quad |I_\pm|^2 = \frac{|I|^2}{N^2},$$

we obtain

$$\begin{aligned}
|I_r^\ell|_{\hat{\sigma}} &= |I|_{\hat{\sigma}} = \frac{|I|^2}{|I|_{\hat{\omega}}} + \frac{|I|^2}{|I|_{\hat{\omega}}} \left\{ \frac{4}{N^2(1-\eta^2)} \right\} + \left\{ \frac{|I_-|^2}{|I_-|_{\hat{\omega}}} \left(\frac{4}{N^2(1-\eta^2)} \right) + \frac{|I_+|^2}{|I_+|_{\hat{\omega}}} \left(\frac{4}{N^2(1-\eta^2)} \right) \right\} + \dots \\
&= \frac{|I|^2}{|I|_{\hat{\omega}}} + \frac{|I|^2}{|I|_{\hat{\omega}}} \left\{ \frac{4}{N^2(1-\eta^2)} \right\} + \left\{ \left(\frac{|I_-|^2}{|I_-|_{\hat{\omega}}} + \frac{|I_+|^2}{|I_+|_{\hat{\omega}}} \right) \left(\frac{4}{N^2(1-\eta^2)} \right) \right\} + \dots \\
&= \frac{|I|^2}{|I|_{\hat{\omega}}} \left[1 + \left\{ \frac{4}{N^2(1-\eta^2)} \right\} + \left\{ \left(\frac{4}{N^2(1-\eta^2)} \right)^2 \right\} + \dots \right] \\
&= \frac{|I|^2}{|I|_{\hat{\omega}}} \left(\frac{1}{1 - \frac{4}{N^2(1-\eta^2)}} \right) = \hat{s}(I) \left(\frac{N^2-1}{N^2-5} \right) = \frac{N^2-1}{N^2-5} |G_r^\ell|_{\hat{\sigma}}.
\end{aligned}$$

□

From the definition $|I_r^\ell|_{\hat{\omega}} |G_r^\ell|_{\hat{\sigma}} = |I_r^\ell|^2$ of $\hat{s}_r^\ell = |G_r^\ell|_{\hat{\sigma}}$ once more, and the inequality $|\pi^{(j)} I_r^\ell|_{\hat{\omega}} \leq \frac{1+\eta}{2} |\pi^{(j+1)} I_r^\ell|_{\hat{\omega}}$, we obtain:

$$\begin{aligned}
(5.4) \quad |G(\pi^{(k)} I_r^\ell)|_{\hat{\sigma}} &= \frac{|\pi^{(k)} I_r^\ell|^2}{|\pi^{(k)} I_r^\ell|_{\hat{\omega}}} \leq \frac{1+\eta}{2} N^2 \frac{|\pi^{(k-1)} I_r^\ell|^2}{|\pi^{(k-1)} I_r^\ell|_{\hat{\omega}}} \\
&\vdots \\
&\leq \left(\frac{1+\eta}{2} N^2 \right)^k \frac{|I_r^\ell|^2}{|I_r^\ell|_{\hat{\omega}}} = \left(\frac{1+\eta}{2} N^2 \right)^k |G(I_r^\ell)|_{\hat{\sigma}}.
\end{aligned}$$

Thus from (5.3), Lemma 2 and (5.4) we have

$$\begin{aligned}
P(I_r^\ell, \hat{\sigma}) &\lesssim \frac{|G(I_r^\ell)|_{\hat{\sigma}}}{|I_r^\ell|} + \frac{1}{N^2} \frac{|G(\pi I_r^\ell)|_{\hat{\sigma}}}{|I_r^\ell|} + \frac{1}{N^4} \frac{|G(\pi^2 I_r^\ell)|_{\hat{\sigma}}}{|I_r^\ell|} + \dots + \frac{1}{N^{2\ell}} \frac{|G(\pi^\ell I_r^\ell)|_{\hat{\sigma}}}{|I_r^\ell|} \\
&\lesssim \frac{|G(I_r^\ell)|_{\hat{\sigma}}}{|I_r^\ell|} + \frac{1}{N^2} \left(\frac{1+\eta}{2} N^2 \right) \frac{|G(I_r^\ell)|_{\hat{\sigma}}}{|I_r^\ell|} + \frac{1}{N^4} \left(\frac{1+\eta}{2} N^2 \right)^2 \frac{|G(I_r^\ell)|_{\hat{\sigma}}}{|I_r^\ell|} \\
&\quad + \dots + \frac{1}{N^{2\ell}} \left(\frac{1+\eta}{2} N^2 \right)^\ell \frac{|G(I_r^\ell)|_{\hat{\sigma}}}{|I_r^\ell|} \\
&\lesssim \frac{|G(I_r^\ell)|_{\hat{\sigma}}}{|I_r^\ell|} \left\{ 1 + \frac{1}{N^2} \left(\frac{1+\eta}{2} N^2 \right) + \frac{1}{N^4} \left(\frac{1+\eta}{2} N^2 \right)^2 + \dots \right\} \\
&= \frac{|G(I_r^\ell)|_{\hat{\sigma}}}{|I_r^\ell|} \left\{ 1 + \left(\frac{1+\eta}{2} \right) + \left(\frac{1+\eta}{2} \right)^2 + \dots \right\} = \frac{|G(I_r^\ell)|_{\hat{\sigma}}}{|I_r^\ell|} \left\{ 1 + \frac{1}{1 - \frac{1+\eta}{2}} \right\},
\end{aligned}$$

and so $P(I_r^\ell, \hat{\sigma}) \approx \frac{|G(I_r^\ell)|_{\hat{\sigma}}}{|I_r^\ell|}$. Thus

$$P(I_r^\ell, \hat{\omega}) P(I_r^\ell, \hat{\sigma}) \approx \frac{|I_r^\ell|_{\hat{\omega}} |G(I_r^\ell)|_{\hat{\sigma}}}{|I_r^\ell|^2} = 1.$$

This verifies the \mathcal{A}_2 condition for the weight pair $(\hat{\sigma}, \hat{\omega})$ (actually we have only verified this for intervals of the form I_r^ℓ , but this is easily seen to be enough since P is a positive operator).

We record the following facts proved implicitly above for future use. Note that (4.10) is used in the first line below.

$$\begin{aligned}
(5.5) \quad P(I_r^\ell, \hat{\omega}) &\approx \frac{|I_r^\ell|_{\hat{\omega}}}{|I_r^\ell|} = \frac{1}{2} \frac{\left(\frac{1+\eta}{2} \right)^{H(I_r^\ell)} \left(\frac{1-\eta}{2} \right)^{T(I_r^\ell)}}{|I_r^\ell|} = \kappa_r^\ell \left(\frac{N}{2} \right)^\ell, \\
P(I_r^\ell, \hat{\sigma}) &\approx \frac{|I_r^\ell|_{\hat{\sigma}}}{|I_r^\ell|} \approx \frac{1}{\kappa_r^\ell} \left(\frac{2}{N} \right)^\ell,
\end{aligned}$$

with

$$(5.6) \quad \kappa_r^\ell \equiv \left(1 + \frac{1}{N}\right)^{H(I_r^\ell)} \left(1 - \frac{1}{N}\right)^{T(I_r^\ell)},$$

and where $I = [0, 1]_\varepsilon$ with $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{d(I)}) \in \{+, -\}^{d(I)}$ and $H(I)$ (for heads) is the number of $+$ signs in $\varepsilon' \equiv (\varepsilon_2, \dots, \varepsilon_{d(I)})$ and $T(I)$ (for tails) is the number of $-$ signs in ε' . See immediately below for a further discussion of the $[0, 1]_\varepsilon$ -notation.

5.2. The forward testing condition. We will see that the forward testing condition for H_b holds with respect to the weight pair $(\hat{\sigma}, \hat{\omega})$ because the measures $\hat{\omega}$ and $\hat{\sigma}$ are self-similar, and the operator H_b is invariant under dilations of scale N . The argument here is considerably more delicate than the corresponding argument in [LaSaUr2], involving some surprising symmetries buried in the redistribution of the measure $\hat{\omega}$. In order to obtain a replicating formula for the measures $\hat{\omega}$ and $\hat{\sigma}$, it is convenient to introduce the notion of a *noncommutative addition* on an embedded dyadic tree.

5.2.1. A noncommutative addition on the dyadic tree. Let $(\mathcal{T}, \succcurlyeq, \mathfrak{r}, \pi, \mathfrak{C})$ denote the dyadic tree with relation \succcurlyeq , root \mathfrak{r} , parent $\pi : \mathcal{T} \setminus \{\mathfrak{r}\} \rightarrow \mathcal{T}$ and children $\mathfrak{C} : \mathcal{T} \rightarrow \mathcal{T} \times \mathcal{T}$ as described earlier. Consider the following three examples of trees defined in terms of our construction of intervals I_r^ℓ above, and with partial order defined in terms of set inclusion:

$$\begin{aligned} \mathcal{D} &= \{I_r^\ell : (\ell, r) \in \mathbb{Z}_+ \times \mathbb{N}, 1 \leq r \leq 2^\ell\} \sim \{(\ell, r) \in \mathbb{Z}_+ \times \mathbb{N} : 1 \leq r \leq 2^\ell\}, \\ \mathcal{D}_- &= \{I_r^\ell : (\ell, r) \in \mathbb{N} \times \mathbb{N}, 1 \leq r \leq 2^{\ell-1}\} \sim \{(\ell, r) \in \mathbb{N} \times \mathbb{N} : 1 \leq r \leq 2^{\ell-1}\}, \\ \mathcal{D}_+ &= \{I_r^\ell : (\ell, r) \in \mathbb{N} \times \mathbb{N}, 2^{\ell-1} < r \leq 2^\ell\} \sim \{(\ell, r) \in \mathbb{N} \times \mathbb{N} : 2^{\ell-1} < r \leq 2^\ell\}. \end{aligned}$$

In order to model dilations and translations of our measures on these dyadic trees, we define a noncommutative addition $\oplus : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ in an embedded tree \mathcal{T} as follows. Fix $\alpha, \beta \in \mathcal{T}$. Then if $\varepsilon \in \{+, -\}^{d(\beta)}$ is the unique string of \pm such that $\beta = (\mathfrak{r})_\varepsilon$, we define $\alpha \oplus \beta = \alpha_\varepsilon$. We can describe this addition more informally when we view \mathcal{T} as embedded in the plane. Indeed, let \mathcal{S} be a copy of \mathcal{T} in the plane and translate (in the plane) the root $\mathfrak{r}_\mathcal{S}$ of \mathcal{S} to lie on top of the point α in the tree \mathcal{T} , and let the remaining points of \mathcal{S} fall on the corresponding points of \mathcal{T} lying below α (for this we ‘dilate’ the translate of the copy \mathcal{S} so as to fit overtop \mathcal{T}). Then the point $\alpha \oplus \beta$ is the point in the tree \mathcal{T} that lies underneath the point β in the tree \mathcal{S} . Finally we define the translation $\beta \oplus \mathcal{T}$ of the tree \mathcal{T} by a point $\beta \in \mathcal{T}$ on the left to be the tree

$$\beta \oplus \mathcal{T} \equiv \{\beta \oplus \alpha : \alpha \in \mathcal{T}\}$$

equipped with the inherited structure from $(\mathcal{T}, \succcurlyeq, \mathfrak{r}, \pi, \mathfrak{C})$. For example, the root of the tree $\beta \oplus \mathcal{T}$ is $\mathfrak{r}(\beta \oplus \mathcal{T}) = \beta$ and the $\beta \oplus \mathcal{T}$ -children of $\beta \oplus \alpha$ are $\beta \oplus \alpha_\pm$.

5.2.2. Self-similarity. We first focus on the left hand child I_1^1 of $I_1^0 = [0, 1]$ and recall that $|I_1^1|_\omega = |I_2^1|_\omega$. To each interval I_r^ℓ we attach the increment $\eta = \eta(I_r^\ell) = \frac{1}{N}$ in the formula (4.3) for the distribution of $\hat{\omega}$ to its two children $I_{r,\text{left}}^\ell = (I_r^\ell)_-$ and $I_{r,\text{right}}^\ell = (I_r^\ell)_+$, i.e. by multiplying with the factors $\frac{1 \pm \eta(I_r^\ell)}{2}$ on the left and right appropriately. Recall from (4.7) that for any interval $I \in \mathcal{D} \setminus \{[0, 1]\}$,

$$\begin{aligned} |I_{--}|_\omega &= \frac{1+\eta}{2} |I_-|_\omega \text{ and } |I_{-+}|_\omega = \frac{1-\eta}{2} |I_-|_\omega, \\ |I_{+-}|_\omega &= \frac{1-\eta}{2} |I_+|_\omega \text{ and } |I_{++}|_\omega = \frac{1+\eta}{2} |I_+|_\omega. \end{aligned}$$

We now show that, as a consequence of $\eta = \eta(I_r^\ell) = \frac{1}{N}$ in Lemma 1, there is a ‘homogeneous’ replicating formula for the self-similar measures $\hat{\omega}_- = \mathbf{1}_{[0, \frac{1}{N}]} \hat{\omega}$ and $\hat{\omega}_+ = \mathbf{1}_{[1-\frac{1}{N}, 1]} \hat{\omega}$, and hence also an ‘inhomogeneous’ replicating formula for both $\hat{\sigma}_- = \mathbf{1}_{[0, \frac{1}{N}]} \hat{\sigma}$ and $\hat{\sigma}_+ = \mathbf{1}_{[1-\frac{1}{N}, 1]} \hat{\sigma}$ with a point mass as the inhomogeneous term. In order to state these replication formulas precisely we need the measures

$$\hat{\omega}_{--} \equiv \mathbf{1}_{[0, \frac{1}{N^2}]} \hat{\omega} \text{ and } \hat{\omega}_{-+} \equiv \mathbf{1}_{[\frac{1}{N} - \frac{1}{N^2}, \frac{1}{N}]} \hat{\omega} \text{ and } \hat{\omega}_{+-} \equiv \mathbf{1}_{[1-\frac{1}{N}, 1-\frac{1}{N} + \frac{1}{N^2}]} \hat{\omega} \text{ and } \hat{\omega}_{++} \equiv \mathbf{1}_{[1-\frac{1}{N^2}, 1]} \hat{\omega},$$

and we define Ref to be reflection about the origin in the real line. For convenience in viewing formulas, we set $\eta = \frac{1}{N}$ in the factors, but retain the notation $\frac{1}{N}$ in dilations and translations.

Let $\text{Dil}_\gamma x \equiv \gamma x$, and $\text{Dil}_\gamma \mu(x) \equiv \frac{1}{\gamma} \mu\left(\text{Dil}_{\frac{1}{\gamma}} x\right) = \frac{1}{\gamma} \mu\left(\frac{x}{\gamma}\right)$ be the measure with the same mass as μ but with support dilated by the positive factor γ . Similarly let $\text{Trans}_\gamma x = x + \gamma$ and $\text{Trans}_\gamma \mu(x) \equiv \mu(\text{Trans}_{-\gamma} x) = \mu(x - \gamma)$. Finally, define $\text{Ref}(x) = -x$ to be reflection about the origin 0 in the real line and $\text{Ref} \mu(x) = \mu(\text{Ref} x) = \mu(-x)$. The following diagram pictures the $\widehat{\omega}$ weight of closed intervals I_r^ℓ in italics, and the $\widehat{\sigma}$ measure of the open intervals G_r^ℓ in bold:

$$\left\{ \begin{array}{l} [\hspace{10em} 1 \hspace{10em}] \\ [\hspace{2em} \frac{1}{2} \hspace{2em}] \text{ (} \hspace{2em} \mathbf{1} \hspace{2em} \text{) } [\hspace{2em} \frac{1}{2} \hspace{2em}] \\ [\hspace{1em} \frac{1+\eta}{4} \hspace{1em}] \text{ (} \hspace{1em} \frac{\mathbf{2}}{\mathbf{N}^2} \hspace{1em} \text{) } [\hspace{1em} \frac{1-\eta}{4} \hspace{1em}] \hspace{10em} [\hspace{1em} \frac{1-\eta}{4} \hspace{1em}] \text{ (} \hspace{1em} \frac{\mathbf{2}}{\mathbf{N}^2} \hspace{1em} \text{) } [\hspace{1em} \frac{1+\eta}{4} \hspace{1em}] \end{array} \right\}.$$

Lemma 3. *We have the replicating formulas for $\widehat{\omega}_-$ and $\widehat{\sigma}_-$;*

$$\begin{aligned} (5.7) \quad \widehat{\omega}_- &= \widehat{\omega}_{--} + \widehat{\omega}_{-+}; \\ \widehat{\omega}_{--} &= \frac{1+\eta}{2} \text{Dil}_{\frac{1}{N}} \widehat{\omega}_- \text{ and } \widehat{\omega}_{-+} = \frac{1-\eta}{2} \text{Trans}_{\frac{1}{N}} \text{Ref Dil}_{\frac{1}{N}} \widehat{\omega}_-, \\ \widehat{\sigma}_- &= \widehat{\sigma}_{--} + \frac{2}{N^2} \delta_{\frac{1}{2N}} + \widehat{\sigma}_{-+}; \\ \widehat{\sigma}_{--} &= \frac{1}{N^2} \frac{2}{1+\eta} \text{Dil}_{\frac{1}{N}} \widehat{\sigma}_- \text{ and } \widehat{\sigma}_{-+} = \frac{1}{N^2} \frac{2}{1-\eta} \text{Trans}_{\frac{1}{N}} \text{Ref Dil}_{\frac{1}{N}} \widehat{\sigma}_-, \end{aligned}$$

and the replicating formulas for $\widehat{\omega}_+$ and $\widehat{\sigma}_+$;

$$\begin{aligned} \widehat{\omega}_+ &= \widehat{\omega}_{+-} + \widehat{\omega}_{++}; \\ \widehat{\omega}_{+-} &= \frac{1-\eta}{2} \text{Trans}_1 \text{Ref Trans}_{\frac{1}{N}} \text{Ref Dil}_{\frac{1}{N}} \text{Ref Trans}_{-1} \widehat{\omega}_+, \\ \widehat{\omega}_{++} &= \frac{1+\eta}{2} \text{Trans}_1 \text{Ref Dil}_{\frac{1}{N}} \text{Ref Trans}_{-1} \widehat{\omega}_+, \\ \widehat{\sigma}_+ &= \widehat{\sigma}_{+-} + \frac{2}{N^2} \delta_{1-\frac{1}{2N}} + \widehat{\sigma}_{++}; \\ \widehat{\sigma}_{+-} &= \frac{1}{N^2} \frac{2}{1-\eta} \text{Trans}_1 \text{Ref Trans}_{\frac{1}{N}} \text{Ref Dil}_{\frac{1}{N}} \text{Ref Trans}_{-1} \widehat{\sigma}_+, \\ \widehat{\sigma}_{++} &= \frac{1}{N^2} \frac{2}{1+\eta} \text{Trans}_1 \text{Ref Dil}_{\frac{1}{N}} \text{Ref Trans}_{-1} \widehat{\sigma}_+, \end{aligned}$$

as well as the dilation invariance of H_b :

$$K_b(x) = \text{Dil}_{\frac{1}{N}} K_b\left(\text{Dil}_{\frac{1}{N}} x\right).$$

Proof. From (4.10), the masses $\lambda(I)$ of the measure $\widehat{\omega}$ on the intervals I in the dyadic tree \mathcal{D} with root $\mathbf{r} = \mathbf{r}(\mathcal{D}) = [0, 1]$ are given by

$$(5.8) \quad \lambda(I) \equiv |I|_{\widehat{\omega}_-} = \frac{1}{2} \left(\frac{1+\eta}{2} \right)^{H(I)} \left(\frac{1-\eta}{2} \right)^{T(I)}$$

where $I = [0, 1]_\varepsilon$ with $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{d(I)}) \in \{+, -\}^{d(I)}$ and $H(I)$ (for heads) is the number of $+$ signs in $\varepsilon' \equiv (\varepsilon_2, \dots, \varepsilon_{d(I)})$ and $T(I)$ (for tails) is the number of $-$ signs in ε' . We can picture the restricted sequence $\lambda : \mathcal{D}_- \rightarrow (0, \infty)$ on the left half of the tree \mathcal{D} in the usual way by displaying the values schematically in an

array as follows:

(5.9)

$$\left[\begin{array}{cccccccc} & & & & & & & \lambda(\mathfrak{r}_-) \\ & & & & & & \swarrow & \searrow \\ & & & & & \lambda(\mathfrak{r}_{--}) & \swarrow & \\ & & & \lambda(\mathfrak{r}_{--}) & \swarrow & \searrow & \lambda(\mathfrak{r}_{--+}) & \\ \lambda(\mathfrak{r}_{----}) & \lambda(\mathfrak{r}_{---}) & \lambda(\mathfrak{r}_{---+}) & \lambda(\mathfrak{r}_{--+}) & \lambda(\mathfrak{r}_{--+}) & \lambda(\mathfrak{r}_{-++}) & \lambda(\mathfrak{r}_{-++}) & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & & & & & & & \\ & \rightarrow & & \searrow & & & & \\ & \rightarrow & & & & \lambda(\mathfrak{r}_{-+}) & & \\ & \rightarrow & & & \swarrow & \searrow & & \\ & \rightarrow & & \lambda(\mathfrak{r}_{-++}) & \lambda(\mathfrak{r}_{-++}) & \lambda(\mathfrak{r}_{-++}) & \lambda(\mathfrak{r}_{-++}) & \lambda(\mathfrak{r}_{-++}) \\ & & & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right]$$

which can be written in shorthand as

(5.10)

$$\left[\begin{array}{cccccccc} & & & & & & & [0] \\ & & & & & & \swarrow & \searrow \\ & & & & & [+] & \swarrow & \\ & & & & \swarrow & \searrow & [+ -] & \\ [+ + +] & [+ +] & [+ + -] & [+ - -] & [+ -] & [+ - +] & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ & & & & & & & \\ & \rightarrow & & \searrow & & & & \\ & \rightarrow & & & & [-] & & \\ & \rightarrow & & & \swarrow & \searrow & & \\ & \rightarrow & & [- -] & [- - -] & [- + -] & [- +] & [- + +] \\ & & & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right]$$

where $[\varepsilon_1 \varepsilon_2 \dots \varepsilon_k] = \left(\frac{1+\varepsilon_2 \eta}{2}\right) \left(\frac{1+\varepsilon_3 \eta}{2}\right) \dots \left(\frac{1+\varepsilon_k \eta}{2}\right)$, i.e. the notation $[- + -]$ stands for the product $\left(\frac{1-\eta}{2}\right) \left(\frac{1+\eta}{2}\right) \left(\frac{1-\eta}{2}\right)$ in (5.8), etc. Note that the $+$'s and $-$'s in square brackets in the second array above refer to the *signs* in front of $\eta = \frac{1}{N}$, while the $+$'s and $-$'s in the first array refer to the *locations* left and right. As a consequence the $+$'s and $-$'s in the two arrays are quite different.

Recall that $\mathcal{D}_- = [0, \frac{1}{N}] \oplus \mathcal{D}$ and define in analogy $\mathcal{D}_{--} = [0, \frac{1}{N^2}] \oplus \mathcal{D}$, etc. The pattern of \pm signs is determined by the formulas in (4.7), and from this we see that the sequences $\{\lambda(I)\}_{I \in \mathcal{D}_-}$ and $\{\lambda(I)\}_{I \in \mathcal{D}_{--}}$

satisfy

$$\begin{aligned}
\{\lambda(I)\}_{I \in \mathcal{D}_{--}} &= \boxed{
\begin{array}{ccccccc}
& & & & & & [+ \\
& & & \swarrow & & \searrow & \\
[+++] & [++] & & & & & [+ -] \\
& & [++ -] & & [+ - -] & & [+ - +] \\
& \vdots & \vdots & & \vdots & & \vdots
\end{array}
} \\
&= \left(\frac{1+\eta}{2} \right) \boxed{
\begin{array}{ccccccc}
& & & & & & [0] \\
& & & \swarrow & \swarrow & \searrow & \\
[++] & & & & & & [-] \\
& \swarrow & \searrow & & & \swarrow & \searrow \\
& [++] & & [+ -] & & [- -] & & [- +] \\
& \vdots & & \vdots & & \vdots & & \vdots
\end{array}
} \\
&= \left(\frac{1+\eta}{2} \right) \{\lambda(I)\}_{I \in \mathcal{D}_-} .
\end{aligned}$$

In fact, we need only check the top two lines in the tree because (4.7) shows that the pattern of signs along *any* row of grandchildren is always $+, -, -, +$, as given in (4.9). The identity $\{\lambda(I)\}_{I \in \mathcal{D}_{--}} = \left(\frac{1+\eta}{2} \right) \{\lambda(I)\}_{I \in \mathcal{D}_-}$ just established between the restrictions of the sequence λ to the trees \mathcal{D}_{--} and \mathcal{D}_- translates precisely into the first formula in the second line of (5.7): $\hat{\omega}_{--} = \frac{1+\eta}{2} \text{Dil}_{\frac{1}{N}} \hat{\omega}_-$.

In order to obtain a similar result for the tree \mathcal{D}_{-+} , we define the operation of reflection Reftree on a sequence $\lambda = \{\lambda(\alpha)\}_{\alpha \in \mathcal{T}}$ by $\text{Reftree } \lambda = \{\lambda(\tilde{\alpha})\}_{\alpha \in \mathcal{T}}$ where if $\alpha = \mathbf{r}_{\epsilon}$ then $\tilde{\alpha} = \mathbf{r}_{-\epsilon}$ where $-\epsilon$ is the sequence of location signs (as in (5.9)) with $+$ and $-$ interchanged - in other words $\text{Reftree } \lambda$ is the mirror image of the sequence λ when presented as an array as in (5.10). Then using that \mathcal{D}_- and \mathcal{D}_{-+} can both be identified

with \mathcal{T} as embedded trees, we have that the sequences $\{\lambda(I)\}_{I \in \mathcal{D}_-}$ and $\{\lambda(I)\}_{I \in \mathcal{D}_{-+}}$ satisfy

(5.11)

$$\begin{aligned}
\{\lambda(I)\}_{I \in \mathcal{D}_{-+}} &= \boxed{\begin{array}{ccccc} & & [-] & & \\ & \swarrow & & \searrow & \\ [- - +] & [- -] & & [- + -] & [- +] \\ \vdots & \vdots & & \vdots & \vdots \end{array}} \\
&= \left(\frac{1-\eta}{2}\right) \boxed{\begin{array}{ccccccc} & & & & [0] & & \\ & & \swarrow & \swarrow & \searrow & \searrow & \\ [- +] & \swarrow & [-] & \searrow & & & [+] \\ \vdots & \vdots & \vdots & \vdots & [+ -] & \swarrow & \searrow \\ & & & & & & [+ +] \\ & & & & \vdots & & \vdots \end{array}} \\
&= \left(\frac{1-\eta}{2}\right) \text{Reftree} \boxed{\begin{array}{ccccccc} & & & & [0] & & \\ & & \swarrow & \swarrow & \searrow & \searrow & \\ [+ +] & \swarrow & [+] & \searrow & & & \\ \vdots & \vdots & \vdots & \vdots & [- -] & \swarrow & \searrow \\ & & & & & & [-] \\ & & & & & & \vdots \\ & & & & & & [- +] \\ & & & & \vdots & & \vdots \end{array}} \\
&= \left(\frac{1-\eta}{2}\right) \text{Reftree} \{\lambda(I)\}_{I \in \mathcal{D}_-} .
\end{aligned}$$

Now for $K \in \mathcal{D}$, and a measure μ supported on the Cantor set $E^{(N)}$ and having K equal to the minimal interval containing the support of μ , we associate to μ the sequence $\{\mu(I)\}_{I \in \mathcal{D}_K}$ of its measures on elements of the tree $\mathcal{D}_K \equiv \{I \in \mathcal{D} : I \subset K\}$; and similarly if μ is supported on $E^{(N)} - 1$, the translate of $E^{(N)}$ by one unit to the left. There is of course a one-to-one correspondence between such measures μ and their associated sequences on the tree \mathcal{D}_K . Then using the identity

$$\{\text{Ref } \mu(I)\}_{I \in \mathcal{D}_{K-1}} = \text{Reftree} \{\mu(I)\}_{I \in \mathcal{D}_K} ,$$

where $\mathcal{D}_K - 1$ denotes the translation of the tree \mathcal{D}_K by one unit to the left, we see that (5.11) translates precisely into the second formula in the second line of (5.7): $\widehat{\omega}_{-+} = \frac{1-\eta}{2} \text{Trans}_{\frac{1}{N}} \text{Ref Dil}_{\frac{1}{N}} \widehat{\omega}_-$.

To obtain the analogous formulas for $\widehat{\omega}_+$ we note that

$$\begin{aligned}
\widehat{\omega}_+ &= \text{Trans}_1 \text{Ref } \widehat{\omega}_- , \\
\widehat{\omega}_{+-} &= \text{Trans}_1 \text{Ref } \widehat{\omega}_{-+} , \\
\widehat{\omega}_{++} &= \text{Trans}_1 \text{Ref } \widehat{\omega}_{--} .
\end{aligned}$$

Thus we have

$$\begin{aligned}
\widehat{\omega}_{+-} &= \text{Trans}_1 \text{Ref } \{\widehat{\omega}_{-+}\} = \text{Trans}_1 \text{Ref } \left\{ \frac{1-\eta}{2} \text{Trans}_{\frac{1}{N}} \text{Ref Dil}_{\frac{1}{N}} [\widehat{\omega}_-] \right\} \\
&= \text{Trans}_1 \text{Ref } \left\{ \frac{1-\eta}{2} \text{Trans}_{\frac{1}{N}} \text{Ref Dil}_{\frac{1}{N}} [\text{Ref Trans}_{-1} \widehat{\omega}_+] \right\} \\
&= \frac{1-\eta}{2} \text{Trans}_1 \text{Ref Trans}_{\frac{1}{N}} \text{Ref Dil}_{\frac{1}{N}} \text{Ref Trans}_{-1} \widehat{\omega}_+
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\omega}_{++} &= \text{Trans}_1 \text{Ref} \{ \widehat{\omega}_{--} \} = \text{Trans}_1 \text{Ref} \left\{ \frac{1+\eta}{2} \text{Dil}_{\frac{1}{N}} [\widehat{\omega}_{-}] \right\} \\
&= \text{Trans}_1 \text{Ref} \left\{ \frac{1+\eta}{2} \text{Dil}_{\frac{1}{N}} [\text{Ref Trans}_{-1} \widehat{\omega}_+] \right\} \\
&= \frac{1+\eta}{2} \text{Trans}_1 \text{Ref Dil}_{\frac{1}{N}} \text{Ref Trans}_{-1} \widehat{\omega}_+ .
\end{aligned}$$

The formulas for $\widehat{\sigma}_\pm$ are proved in the same way as for $\widehat{\omega}_\pm$ using Lemma 2 and the definition (4.11) of the weights $\widehat{s}_r^\ell = |G_r^\ell|_{\widehat{\sigma}}$ in $\widehat{\sigma}$ in terms of $\widehat{\omega}$:

$$|I_r^\ell|_{\widehat{\sigma}} = \frac{N^2 - 1}{N^2 - 5} |G_r^\ell|_{\widehat{\sigma}} = \frac{N^2 - 1}{N^2 - 5} \frac{|I_r^\ell|^2}{|I_r^\ell|_{\widehat{\omega}}}.$$

The straightforward details are left to the reader, and this completes the proof of Lemma 3. \square

5.2.3. *Completion of the proof.* Armed with Lemma 3 we now prove the forward testing condition for the interval $I_0^1 = [0, 1]$:

$$\begin{aligned}
(5.12) \quad \int |H_b \widehat{\sigma}|^2 \widehat{\omega} &= \int |H_b (\widehat{\sigma}_- + \delta_{\frac{1}{2}} + \widehat{\sigma}_+)|^2 \widehat{\omega}_- + \int |H_b (\widehat{\sigma}_- + \delta_{\frac{1}{2}} + \widehat{\sigma}_+)|^2 \widehat{\omega}_+ \\
&\leq (1 + \varepsilon) \left\{ \int |H_b \widehat{\sigma}_-|^2 \widehat{\omega}_- + \int |H_b \widehat{\sigma}_+|^2 \widehat{\omega}_+ \right\} + \mathcal{R}_\varepsilon,
\end{aligned}$$

where the remainder term \mathcal{R}_ε is easily seen to satisfy

$$\mathcal{R}_\varepsilon \lesssim_\varepsilon \mathcal{A}_2 \left(\int \dot{\sigma} \right),$$

since the supports of $\delta_{\frac{1}{2}} + \widehat{\sigma}_+$ and $\widehat{\omega}_-$ are well separated, as are those of $\widehat{\sigma}_- + \delta_{\frac{1}{2}}$ and $\widehat{\omega}_+$. Indeed, we first use $(a + b)^2 \leq (1 + \varepsilon) a^2 + (1 + \frac{1}{\varepsilon}) b^2$ to obtain

$$\begin{aligned}
&\int |H_b (\widehat{\sigma}_- + \delta_{\frac{1}{2}} + \widehat{\sigma}_+)|^2 \widehat{\omega}_- \\
&\leq \int \left(|H_b (\widehat{\sigma}_-)| + |H_b (\delta_{\frac{1}{2}} + \widehat{\sigma}_+)| \right)^2 \widehat{\omega}_- \\
&\leq \int \left\{ (1 + \varepsilon) |H_b (\widehat{\sigma}_-)|^2 + \left(1 + \frac{1}{\varepsilon} \right) |H_b (\delta_{\frac{1}{2}} + \widehat{\sigma}_+)|^2 \right\} \widehat{\omega}_-,
\end{aligned}$$

and then for example, since $K_b(y - x) = 1$ for $x \in [0, \frac{1}{N}]$ and $y \in [\frac{N-1}{N}, 1]$,

$$\begin{aligned}
\int |H_b (\widehat{\sigma}_+)|^2 \widehat{\omega}_- &= \int_{[0, \frac{1}{N}]} \left| \int_{[\frac{N-1}{N}, 1]} d\widehat{\sigma}_+(y) \right|^2 d\widehat{\omega}_- \\
&= \left| \left[1 - \frac{1}{N}, 1 \right] \right|_{\widehat{\sigma}_+}^2 | [0, \frac{1}{M}] |_{\widehat{\omega}_-} \\
&\leq \frac{|[0, 1]|_{\widehat{\sigma}} |[0, 1]|_{\widehat{\omega}}}{|[0, 1]|^2} \int \widehat{\sigma}_+ \leq \mathcal{A}_2 \int \widehat{\sigma}_+ .
\end{aligned}$$

Now we continue to the grandchildren so as to exploit the replication formulas. We write

$$\begin{aligned}
\int |H_b \widehat{\sigma}_-|^2 \widehat{\omega}_- &= \int |H_b \left(\widehat{\sigma}_{--} + \frac{2}{N^2} \delta_{\frac{1}{2N}} + \widehat{\sigma}_{-+} \right)|^2 \widehat{\omega}_{--} + \int |H_b \left(\widehat{\sigma}_{--} + \frac{2}{N^2} \delta_{\frac{1}{2N}} + \widehat{\sigma}_{-+} \right)|^2 \widehat{\omega}_{-+} \\
&\leq (1 + \varepsilon) \left\{ \int |H_b \widehat{\sigma}_{--}|^2 \widehat{\omega}_{--} + \int |H_b \widehat{\sigma}_{-+}|^2 \widehat{\omega}_{-+} \right\} + \mathcal{R}_\varepsilon,
\end{aligned}$$

where again the remainder term \mathcal{R}_ε is easily seen just as above to satisfy

$$\mathcal{R}_\varepsilon \lesssim_\varepsilon \mathcal{A}_2 \left(\int \widehat{\sigma}_- \right),$$

since the supports of $\frac{2}{N^2}\delta_{\frac{1}{2N}} + \hat{\sigma}_{-+}$ and $\hat{\omega}_{--}$ are well separated, as are those of $\hat{\sigma}_{--} + \frac{2}{N^2}\delta_{\frac{1}{2N}}$ and $\hat{\omega}_{-+}$. Similarly we have

$$\int |H_b \hat{\sigma}_+|^2 \hat{\omega}_+ \leq (1 + \varepsilon) \left\{ \int |H_b \hat{\sigma}_{+-}|^2 \hat{\omega}_{+-} + \int |H_b \hat{\sigma}_{++}|^2 \hat{\omega}_{++} \right\} + C\mathcal{A}_2 \left(\int \hat{\sigma}_+ \right).$$

But now we note that

$$\begin{aligned} \int |H_b \hat{\sigma}_{--}|^2 \hat{\omega}_{--} &= \int |H_b \hat{\sigma}_{--}(x)|^2 \frac{1+\eta}{2} \text{Dil}_{\frac{1}{N}} \hat{\omega}_{--}(x) = \frac{1+\eta}{2} \int |H_b \hat{\sigma}_{--}(x)|^2 N \hat{\omega}_{--}(Nx) \\ &= \frac{1+\eta}{2} \int |H_b \hat{\sigma}_{--}\left(\frac{y}{N}\right)|^2 N \hat{\omega}_{--}(y) \frac{dx}{dy} \quad (y = Nx) \\ &= \frac{1+\eta}{2} \int \left| \int K_b\left(\frac{y}{N} - z\right) \hat{\sigma}_{--}(z) \right|^2 \hat{\omega}_{--}(y) \\ &= \frac{1+\eta}{2} \int \left| \int K_b\left(\frac{y}{N} - z\right) \frac{2}{1+\eta} \frac{1}{N^2} \text{Dil}_{\frac{1}{N}} \hat{\sigma}_{--}(z) \right|^2 \hat{\omega}_{--}(y) \\ &= \frac{2}{1+\eta} \frac{1}{N^4} \int \left| \int K_b\left(\frac{y}{N} - z\right) N \hat{\sigma}_{--}(Nz) \right|^2 \hat{\omega}_{--}(y) \\ &= \frac{2}{1+\eta} \frac{1}{N^4} \int \left| \int K_b\left(\frac{y}{N} - \frac{u}{N}\right) N \hat{\sigma}_{--}(u) \frac{dz}{du} \right|^2 \hat{\omega}_{--}(y) \quad (u = Nz) \\ &= \frac{2}{1+\eta} \frac{1}{N^4} \int \left| \int N K_b(y - u) \hat{\sigma}_{--}(u) \right|^2 \hat{\omega}_{--}(y) = \frac{2}{1+\eta} \frac{1}{N^2} \int |H_b \hat{\sigma}_{--}|^2 \hat{\omega}_{--}, \end{aligned}$$

and similarly

$$|H_b \hat{\sigma}_{-+}|^2 \hat{\omega}_{-+} = \frac{2}{1-\eta} \frac{1}{N^2} \int |H_b \hat{\sigma}_{-}|^2 \hat{\omega}_{-}.$$

Moreover, we also have from trivial modifications of the same arguments that

$$\int |H_b \hat{\sigma}_{+-}|^2 \hat{\omega}_{+-} = \frac{2}{1-\eta} \frac{1}{N^2} \int |H_b \hat{\sigma}_+|^2 \hat{\omega}_+ \text{ and } \int |H_b \hat{\sigma}_{++}|^2 \hat{\omega}_{++} = \frac{2}{1+\eta} \frac{1}{N^2} \int |H_b \hat{\sigma}_+|^2 \hat{\omega}_+.$$

Altogether we have that

$$\begin{aligned} &\int |H_b \hat{\sigma}_{--}|^2 \hat{\omega}_{--} + \int |H_b \hat{\sigma}_+|^2 \hat{\omega}_+ \\ &\leq (1 + \varepsilon) \left\{ \int |H_b \hat{\sigma}_{--}|^2 \hat{\omega}_{--} + \int |H_b \hat{\sigma}_{-+}|^2 \hat{\omega}_{-+} + \int |H_b \hat{\sigma}_{+-}|^2 \hat{\omega}_{+-} + \int |H_b \hat{\sigma}_{++}|^2 \hat{\omega}_{++} \right\} + C\mathcal{A}_2^2 \left(\int \hat{\sigma} \right) \\ &= (1 + \varepsilon) \left(\frac{4}{1-\eta} + \frac{4}{1+\eta} \right) \frac{1}{N^2} \left\{ \int |H_b \hat{\sigma}_{--}|^2 \hat{\omega}_{--} + \int |H_b \hat{\sigma}_+|^2 \hat{\omega}_+ \right\} + C\mathcal{A}_2 \left(\int \hat{\sigma} \right), \end{aligned}$$

and provided both $\int |H_b \hat{\sigma}_{--}|^2 \hat{\omega}_{--}$ and $\int |H_b \hat{\sigma}_+|^2 \hat{\omega}_+$ are finite we conclude that

$$\int |H_b \hat{\sigma}_{--}|^2 \hat{\omega}_{--} + \int |H_b \hat{\sigma}_+|^2 \hat{\omega}_+ \leq \frac{1}{1 - \left(\frac{2}{1+\eta} + \frac{2}{1-\eta} \right) \frac{1+\varepsilon}{N^2}} C\mathcal{A}_2^2 \left(\int \hat{\sigma} \right)$$

for $1 - \left(\frac{4}{1+\eta} + \frac{4}{1-\eta} \right) \frac{1+\varepsilon}{N^2} > 0$, e.g. if $0 < \varepsilon < 1$ and $N \geq 5$. Finally then we have

$$\int |H_b \hat{\sigma}|^2 \hat{\omega} \leq (1 + \varepsilon) \left\{ \int |H_b \hat{\sigma}_{--}|^2 \hat{\omega}_{--} + \int |H_b \hat{\sigma}_+|^2 \hat{\omega}_+ \right\} + C\mathcal{A}_2 \left(\int \hat{\sigma} \right) \leq C\mathcal{A}_2 \left(\int \hat{\sigma} \right).$$

To avoid making the assumption that $\int |H\hat{\sigma}|^2 \hat{\omega}$ is finite, we use the approximations

$$(5.13) \quad \begin{aligned} d\hat{\omega}^{(m)}(x) &= \sum_{i=1}^{2^m} 2^{-m} \kappa_i^m \frac{1}{|I_i^m|} \mathbf{1}_{I_i^m}(x) dx, \\ \hat{\sigma}^{(n)} &= \sum_{k < n} \sum_{j=1}^{2^k} \hat{s}_j^k \delta_{z_j^k}, \end{aligned}$$

(κ_i^m is defined in (5.6)) as in the argument for the forward testing condition for the corresponding weight pair in [LaSaUr2]. Note that $\hat{\omega}^{(m)} = \omega_m$ is the m^{th} generation redistribution of ω as defined in Subsection 4.1, and that $\hat{\sigma}^{(n)}$ is a partial sum of the series defining $\hat{\sigma}$. Then for fixed $n \leq m$ we obtain in analogy with [LaSaUr2] that

$$\int |H_b \hat{\sigma}^{(n)}|^2 \hat{\omega}^{(m)} \leq C\mathcal{A}_2 \left(\int \hat{\sigma}^{(n)} \right) + (1 - \delta)^{m-n} \int |H_b \hat{\sigma}^{(0)}|^2 \hat{\omega}^{(m-n)},$$

where $\int |H_b \hat{\sigma}^{(0)}|^2 \hat{\omega}^{(m-n)} \leq C\mathcal{A}_2$ independent of m and n . Now for fixed n we let $m \rightarrow \infty$ which gives

$$\int |H_b \hat{\sigma}^{(n)}|^2 \hat{\omega} \leq C\mathcal{A}_2 \left(\int \hat{\sigma}^{(n)} \right)$$

since on the support of $\hat{\omega}^{(n)}$ the function $H_b \hat{\sigma}^{(n)}$ is continuous. Then let $n \rightarrow \infty$ to obtain using Fatou's lemma,

$$\int |H_b \hat{\sigma}|^2 \hat{\omega} = \int \liminf_{n \rightarrow \infty} |H_b \hat{\sigma}^{(n)}|^2 \hat{\omega} \leq \liminf_{n \rightarrow \infty} \int |H_b \hat{\sigma}^{(n)}|^2 \hat{\omega} \leq C\mathcal{A}_2 \liminf_{n \rightarrow \infty} \left(\int \hat{\sigma}^{(n)} \right) = C\mathcal{A}_2 \left(\int \hat{\sigma} \right),$$

since for any x in the Cantor set $E^{(N)}$, $\lim_{n \rightarrow \infty} H_b \hat{\sigma}^{(n)}(x) = H_b \hat{\sigma}(x)$ because the support of $\hat{\sigma}$ consists of points z_r^ℓ whose distances from x are at least $\frac{c}{N^\ell}$.

This completes the proof of the forward testing condition for the interval $I_0^1 = [0, 1]$. The arguments above are easily adapted to prove the forward testing condition for any $I_r^\ell \in \mathcal{D}$ not equal to I_0^1 . Indeed, the only essential difference, apart from scaling factors related to ℓ , is that, unlike the case $I = I_0^1$, there is a redistribution of the $\hat{\omega}$ -measures of the children of I_r^ℓ . Consequently there are recursion relations for the children of I_r^ℓ as opposed to the grandchildren as for the case $I_0^1 = [0, 1]$, resulting in a small simplification of the proof. We leave the straightforward details for the reader.

It thus remains only to establish the forward testing condition for an arbitrary interval I contained in $[0, 1]$. The analogous estimate for the forward testing condition in [LaSaUr2] is actually more delicate than indicated there, where it was claimed that "the general case now follows without much extra work", and consequently we will give a detailed proof here.

So let $I = [a, b]$ be an arbitrary *closed* subinterval of $[0, 1]$. Then there is a unique point z_r^ℓ such that $\hat{s}_r^\ell = \sup_{z_j^k \in I} \hat{s}_j^k$. Then $|I \setminus I_r^\ell|_{\hat{\sigma}} = 0$ follows from the choice of z_r^ℓ and the fact that I is closed. Thus from Lemma 2 we now obtain

$$|I|_{\hat{\sigma}} \leq |I_r^\ell|_{\hat{\sigma}} = \frac{N^2 - 1}{N^2 - 5} |G_r^\ell|_{\hat{\sigma}} = \frac{N^2 - 1}{N^2 - 5} |\{z_r^\ell\}|_{\hat{\sigma}} \leq \frac{N^2 - 1}{N^2 - 5} |I|_{\hat{\sigma}},$$

which shows that $|I|_{\hat{\sigma}} \approx |\{z_r^\ell\}|_{\hat{\sigma}} = \hat{s}_r^\ell$. We set

$$I_{\text{left}} = [a, z_r^\ell) \text{ and } I_{\text{right}} = (z_r^\ell, b].$$

Then we estimate

$$\begin{aligned} \int_I |H_b(\mathbf{1}_I \hat{\sigma})|^2 \hat{\omega} &\leq 3 \int_I |H_b(\mathbf{1}_{I_{\text{left}}} \hat{\sigma})|^2 \hat{\omega} + 3 \int_I |H_b(\hat{s}_r^\ell \delta_{z_r^\ell})|^2 \hat{\omega} + 3 \int_I |H_b(\mathbf{1}_{I_{\text{right}}} \hat{\sigma})|^2 \hat{\omega} \\ &\leq 3 \int_{I_r^\ell} |H_b(\mathbf{1}_{I_{\text{left}}} \hat{\sigma})|^2 \hat{\omega} + 3 \int_{I_r^\ell} |H_b(\hat{s}_r^\ell \delta_{z_r^\ell})|^2 \hat{\omega} + 3 \int_{I_r^\ell} |H_b(\mathbf{1}_{I_{\text{right}}} \hat{\sigma})|^2 \hat{\omega} \end{aligned}$$

where the second line holds since $|I \setminus I_r^\ell|_{\widehat{\omega}} = 0$ follows from the choice of \dot{z}_r^ℓ . Since the middle term is trivially estimated by $A_2 \widehat{s}_r^\ell$, it suffices by symmetry to prove that

$$(5.14) \quad \int_{I_r^\ell} \left| H_b \left(\mathbf{1}_{I_{\text{left}}} \widehat{\sigma} \right) \right|^2 \widehat{\omega} \lesssim \widehat{s}_r^\ell.$$

Now let $\mathcal{M}_{I_{\text{left}}} \equiv \{J \in \mathcal{D} : J \subset I_{\text{left}} \text{ is maximal}\}$ consist of the maximal intervals from \mathcal{D} that lie inside I_{left} . Then we can order the intervals in $\mathcal{M}_{I_{\text{left}}}$ from right to left as $\{M_i\}_{i=1}^\infty$ where $M_i = I_{r_i}^{\ell_i}$ for a *strictly* increasing sequence $\{\ell_i\}_{i=1}^\infty$ with $\ell_1 > \ell$ (of course the sequence could be finite but then the proof is similar with obvious modifications) and a choice of r_i so that M_{i+1} lies to the left of M_i for all $i \geq 1$. If $(I_r^\ell)_-$ is contained in I_{left} , then $(I_r^\ell)_-$ is the only maximal interval M_1 and the estimate (5.14) is then trivial. So we assume from now on that $(I_r^\ell)_- \not\subset I_{\text{left}}$. Then the sequence $\{M_i\}_{i=1}^\infty$ is defined inductively as follows.

- (1) Define $J \equiv (I_r^\ell)_-$. If $J_+ \subset I_{\text{left}}$ then we set $M_1 = J_+$. If $J_+ \not\subset I_{\text{left}}$ but $J_{++} \subset I_{\text{left}}$, then we set $M_1 = J_{++}$. Otherwise, we continue in this way and let $M_1 = J \underbrace{\quad}_{++ \dots +}^{m_1}$ where $J \underbrace{\quad}_{++ \dots +}^{m_1}$ with $m_1 \geq 1$ is the first such interval that is contained in I_{left} .
- (2) Define K to be the sibling of $M_1 = J \underbrace{\quad}_{++ \dots +}^{m_1}$ in the tree \mathcal{D} . Note that $K \not\subset I_{\text{left}}$ by the choice of M_1 . If $K_+ \subset I_{\text{left}}$ then we set $M_2 = K_+$. If $K_+ \not\subset I_{\text{left}}$ but $K_{++} \subset I_{\text{left}}$, then we set $M_2 = K_{++}$. Otherwise, we continue in this way and let $M_2 = K \underbrace{\quad}_{++ \dots +}^{m_2}$ where $K \underbrace{\quad}_{++ \dots +}^{m_2}$ with $m_2 \geq 1$ is the first such interval that is contained in I_{left} .
- (3) Define L to be the sibling of $K \underbrace{\quad}_{++ \dots +}^{m_2}$ in the tree \mathcal{D} . Note that $L \not\subset I_{\text{left}}$ by the choice of M_2 . If $L_+ \subset I_{\text{left}}$ then we set $M_3 = L_+$. If $L_+ \not\subset I_{\text{left}}$ but $L_{++} \subset I_{\text{left}}$, then we set $M_3 = L_{++}$. Otherwise, we continue in this way and let $M_3 = L \underbrace{\quad}_{++ \dots +}^{m_3}$ where $L \underbrace{\quad}_{++ \dots +}^{m_3}$ with $m_3 \geq 1$ is the first such interval that is contained in I_{left} .
- (4) Continue to define M_4, M_5, M_6, \dots in this way until the procedure either terminates in a finite sequence $\{M_i\}_{i=1}^N$ or defines an infinite sequence $\{M_i\}_{i=1}^\infty$.

Then we estimate

$$\begin{aligned} \int_{I_r^\ell} \left| H_b \left(\mathbf{1}_{I_{\text{left}}} \widehat{\sigma} \right) \right|^2 \widehat{\omega} &= \int_{I_r^\ell} \left| H_b \left(\sum_{i=1}^\infty \mathbf{1}_{M_i} \widehat{\sigma} \right) \right|^2 \widehat{\omega} = \sum_{i,j=1}^\infty \int_{I_r^\ell} H_b \left(\mathbf{1}_{M_i} \widehat{\sigma} \right) \overline{H_b \left(\mathbf{1}_{M_j} \widehat{\sigma} \right)} \widehat{\omega} \\ &\leq 2 \sum_{i=1}^\infty \sum_{j=i}^\infty \left| \int_{I_r^\ell} H_b \left(\mathbf{1}_{M_i} \widehat{\sigma} \right) \overline{H_b \left(\mathbf{1}_{M_j} \widehat{\sigma} \right)} \widehat{\omega} \right| \\ &\leq 2 \sum_{i=1}^\infty \sum_{j=i}^\infty \sqrt{\int_{I_r^\ell} \left| H_b \left(\mathbf{1}_{M_i} \widehat{\sigma} \right) \right|^2 \widehat{\omega}} \sqrt{\int_{I_r^\ell} \left| H_b \left(\mathbf{1}_{M_j} \widehat{\sigma} \right) \right|^2 \widehat{\omega}} \\ &\lesssim \sum_{i=1}^\infty \sum_{j=i}^\infty \left\{ \frac{1}{(j-i+1)^2} \int_{I_r^\ell} \left| H_b \left(\mathbf{1}_{M_i} \widehat{\sigma} \right) \right|^2 \widehat{\omega} + (j-i+1)^2 \int_{I_r^\ell} \left| H_b \left(\mathbf{1}_{M_j} \widehat{\sigma} \right) \right|^2 \widehat{\omega} \right\} \\ &\lesssim \sum_{i=1}^\infty \int_{I_r^\ell} \left| H_b \left(\mathbf{1}_{M_i} \widehat{\sigma} \right) \right|^2 \widehat{\omega} + \sum_{j=1}^\infty j^3 \int_{I_r^\ell} \left| H_b \left(\mathbf{1}_{M_j} \widehat{\sigma} \right) \right|^2 \widehat{\omega} \\ &\lesssim \sum_{k=1}^\infty k^3 \int_{I_r^\ell} \left| H_b \left(\mathbf{1}_{M_k} \widehat{\sigma} \right) \right|^2 \widehat{\omega} = \sum_{k=1}^\infty k^3 \int_{I_{r_k}^\ell} \left| H_b \left(\mathbf{1}_{I_{r_k}^\ell} \widehat{\sigma} \right) \right|^2 \widehat{\omega}. \end{aligned}$$

Now we note that if $I, J \in \mathcal{D}$ are disjoint, then $2I \cap J = \emptyset$ as well. Thus from the testing condition for intervals $I \in \mathcal{D}$ that was proved above, together with the \mathcal{A}_2 condition, we easily obtain the *full* testing

condition for the intervals $I \in \mathcal{D}$:

$$\begin{aligned} \int_0^1 |H_b(\mathbf{1}_I \hat{\sigma})|^2 \hat{\omega} &= \int_I |H_b(\mathbf{1}_I \hat{\sigma})|^2 \hat{\omega} + \int_{[0,1] \setminus 2I} |H_b(\mathbf{1}_I \hat{\sigma})|^2 \hat{\omega} \\ &\leq \mathfrak{T}_{H_b} |I|_{\hat{\sigma}} + C_N \mathcal{A}_2 |I|_{\hat{\sigma}} \lesssim |I|_{\hat{\sigma}} . \end{aligned}$$

Thus using $\int_{I_r^\ell} |H_b(\mathbf{1}_{I_{r_k}^{\ell_k}} \hat{\sigma})|^2 \hat{\omega} \lesssim |I_{r_k}^{\ell_k}|_{\hat{\sigma}}$ we obtain

$$\int_{I_r^\ell} |H_b(\mathbf{1}_{I_{\text{left}} \hat{\sigma}})|^2 \hat{\omega} \lesssim \sum_{k=1}^{\infty} k^3 \int_{I_r^\ell} |H_b(\mathbf{1}_{I_{r_k}^{\ell_k}} \hat{\sigma})|^2 \hat{\omega} \lesssim \sum_{k=1}^{\infty} k^3 |I_{r_k}^{\ell_k}|_{\hat{\sigma}} ,$$

and it remains to show that

$$\sum_{k=1}^{\infty} k^3 |I_{r_k}^{\ell_k}|_{\hat{\sigma}} \lesssim |I_r^\ell|_{\hat{\sigma}} .$$

But this latter inequality is an easy consequence of the geometric decay of the numbers $|I_{r_k}^{\ell_k}|_{\hat{\sigma}}$ as $k \rightarrow \infty$. Indeed, returning to the inductive definition of $M_k = I_{r_k}^{\ell_k}$, we see from the inequality $|I_{\pm}|_{\hat{\sigma}} \leq \frac{2}{N^2(1-\eta)} |I|_{\hat{\sigma}}$ that

$$\begin{aligned} (1) \quad |M_1|_{\hat{\sigma}} &= \left| J \underbrace{+ \dots +}_{m_1} \right|_{\hat{\sigma}} \leq \left(\frac{2}{N^2(1-\eta)} \right)^{m_1} |J|_{\hat{\sigma}} ; \\ (2) \quad |K|_{\hat{\sigma}} &\leq \frac{1+\eta}{1-\eta} |M_1|_{\hat{\sigma}} \text{ and } |M_2|_{\hat{\sigma}} = \left| K \underbrace{+ \dots +}_{m_2} \right|_{\hat{\sigma}} \leq \left(\frac{2}{N^2(1-\eta)} \right)^{m_2} |K|_{\hat{\sigma}} ; \\ (3) \quad |L|_{\hat{\sigma}} &\leq \frac{1+\eta}{1-\eta} |M_2|_{\hat{\sigma}} \text{ and } |M_3|_{\hat{\sigma}} = \left| L \underbrace{+ \dots +}_{m_3} \right|_{\hat{\sigma}} \leq \left(\frac{2}{N^2(1-\eta)} \right)^{m_3} |L|_{\hat{\sigma}} ; \\ (4) \quad &\text{etc.} \end{aligned}$$

Thus we see that

$$\begin{aligned} |M_1|_{\hat{\sigma}} &\leq \left(\frac{2}{N^2(1-\eta)} \right)^{m_1} |I_r^\ell|_{\hat{\sigma}} , \\ |M_2|_{\hat{\sigma}} &\leq \left(\frac{2}{N^2(1-\eta)} \right)^{m_2} \frac{1+\eta}{1-\eta} |M_1|_{\hat{\sigma}} , \\ |M_3|_{\hat{\sigma}} &\leq \left(\frac{2}{N^2(1-\eta)} \right)^{m_3} \frac{1+\eta}{1-\eta} |M_2|_{\hat{\sigma}} , \\ &\text{etc.} \end{aligned}$$

and so

$$\begin{aligned} |I_{r_k}^{\ell_k}|_{\hat{\sigma}} &= |M_k|_{\hat{\sigma}} \leq \left(\frac{2}{N^2(1-\eta)} \right)^{m_1+\dots+m_k} \left(\frac{1+\eta}{1-\eta} \right)^{k-1} |I_r^\ell|_{\hat{\sigma}} \\ &\leq \left(\frac{2}{N^2(1-\eta)} \frac{1+\eta}{1-\eta} \right)^k |I_r^\ell|_{\hat{\sigma}} . \end{aligned}$$

Thus we have

$$\sum_{k=1}^{\infty} k^3 |I_{r_k}^{\ell_k}|_{\hat{\sigma}} \leq \sum_{k=1}^{\infty} k^3 \left(\frac{2}{N^2(1-\eta)} \frac{1+\eta}{1-\eta} \right)^k |I_r^\ell|_{\hat{\sigma}} \leq C_N |I_r^\ell|_{\hat{\sigma}} ,$$

provided $\frac{2(1+\frac{1}{N})}{N^2(1-\frac{1}{N})^2} < 1$, i.e. $N \geq 3$, and this completes the proof of the forward testing condition for the weight pair $(\hat{\sigma}, \hat{\omega})$ relative to H_b .

5.3. The backward testing condition. The backward testing condition will hold for the weight pair $(\hat{\sigma}, \hat{\omega})$ because we have arranged that $H_b \hat{\omega}(\dot{z}_j^k) = 0$. Indeed, for an interval I_r^ℓ with $\dot{z}_j^k \in I_r^\ell$, we claim that

$$(5.15) \quad |H_b(\mathbf{1}_{I_r^\ell} \hat{\omega})(\dot{z}_j^k)| \lesssim P(I_r^\ell, \hat{\omega}).$$

To see (5.15) let $I_s^{\ell-1}$ denote the parent of I_r^ℓ and let I_{r+1}^ℓ denote the other child of $I_s^{\ell-1}$. Then we have using $H_b \hat{\omega}(\dot{z}_j^k) = 0$,

$$\begin{aligned} H_b(\mathbf{1}_{I_r^\ell} \hat{\omega})(\dot{z}_j^k) &= -H_b(\mathbf{1}_{(I_s^{\ell-1})^c} \hat{\omega})(\dot{z}_j^k) \\ &= -H_b(\mathbf{1}_{(I_s^{\ell-1})^c} \hat{\omega})(\dot{z}_j^k) - H_b(\mathbf{1}_{I_{r+1}^\ell} \hat{\omega})(\dot{z}_j^k). \end{aligned}$$

Using $H_b \hat{\omega}(\dot{z}_r^\ell) = 0$ we also have that

$$\begin{aligned} H_b(\mathbf{1}_{(I_s^{\ell-1})^c} \hat{\omega})(\dot{z}_j^k) &= H_b(\mathbf{1}_{(I_s^{\ell-1})^c} \hat{\omega})(\dot{z}_r^\ell) - \left\{ H_b(\mathbf{1}_{(I_s^{\ell-1})^c} \hat{\omega})(\dot{z}_r^\ell) - H_b(\mathbf{1}_{(I_s^{\ell-1})^c} \hat{\omega})(\dot{z}_j^k) \right\} \\ &= -H_b(\mathbf{1}_{I_s^{\ell-1}} \hat{\omega})(\dot{z}_r^\ell) - A, \end{aligned}$$

where

$$A \equiv H_b(\mathbf{1}_{(I_s^{\ell-1})^c} \hat{\omega})(\dot{z}_r^\ell) - H_b(\mathbf{1}_{(I_s^{\ell-1})^c} \hat{\omega})(\dot{z}_j^k).$$

Combining equalities yields

$$H_b(\mathbf{1}_{I_r^\ell} \hat{\omega})(\dot{z}_j^k) = H_b(\mathbf{1}_{I_s^{\ell-1}} \hat{\omega})(\dot{z}_r^\ell) + A - H_b(\mathbf{1}_{I_{r+1}^\ell} \hat{\omega})(\dot{z}_j^k).$$

We then have from (5.5) that for (k, j) such that $\dot{z}_j^k \in I_r^\ell$,

$$\begin{aligned} |H_b(\mathbf{1}_{I_s^{\ell-1}} \hat{\omega})(\dot{z}_r^\ell)| &\lesssim \frac{|I_s^{\ell-1}|_{\hat{\omega}}}{|I_s^{\ell-1}|} \approx P(I_s^{\ell-1}, \hat{\omega}), \\ |A| &\lesssim \int_{(I_s^{\ell-1})^c} \left| \frac{1}{x - \dot{z}_r^\ell} - \frac{1}{x - \dot{z}_j^k} \right| d\hat{\omega}(x) \lesssim \int_{(I_s^{\ell-1})^c} \frac{|I_r^\ell|}{|x - \dot{z}_s^{\ell-1}|^2} d\hat{\omega}(x) \lesssim P(I_r^\ell, \hat{\omega}), \\ |H_b(\mathbf{1}_{I_{r+1}^\ell} \hat{\omega})(\dot{z}_j^k)| &\lesssim \frac{|I_s^{\ell-1}|_{\hat{\omega}}}{|I_s^{\ell-1}|} \approx P(I_s^{\ell-1}, \hat{\omega}), \end{aligned}$$

which proves (5.15) since $P(I_s^{\ell-1}, \hat{\omega}) \approx P(I_r^\ell, \hat{\omega})$.

Now, using (5.15) and the estimate $P(I_r^\ell, \hat{\omega}) \approx \frac{|I_r^\ell|_{\hat{\omega}}}{|I_r^\ell|}$, we compute that

$$\begin{aligned} (5.16) \quad \int_{I_r^\ell} |H_b(\mathbf{1}_{I_r^\ell} \hat{\omega})|^2 d\hat{\sigma} &= \sum_{(k,j): \dot{z}_j^k \in I_r^\ell} |H_b(\mathbf{1}_{I_r^\ell} \hat{\omega})(\dot{z}_j^k)|^2 \hat{s}_j^k \leq C \sum_{(k,j): \dot{z}_j^k \in I_r^\ell} |P(I_r^\ell, \hat{\omega})|^2 \hat{s}_j^k \\ &\lesssim |I_r^\ell|_{\hat{\sigma}} \left(\frac{|I_r^\ell|_{\hat{\omega}}}{|I_r^\ell|} \right)^2 \lesssim \mathcal{A}_2 |I_r^\ell|_{\hat{\omega}}. \end{aligned}$$

This proves the case $I = I_r^\ell$ of the backward testing condition (1.6) for the weight pair $(\hat{\sigma}, \hat{\omega})$. The general case of an arbitrary interval follows from this using the argument at the end of Subsubsection 5.2.3, that was used to obtain the general forward testing condition from the case $I = I_r^\ell$.

5.4. The weak boundedness property. Here we prove the weak boundedness property for the weight pair $(\hat{\sigma}, \hat{\omega})$ relative to the flattened Hilbert transform H_b , i.e.

$$\left| \int_J H_b(\mathbf{1}_I \hat{\sigma}) d\hat{\omega} \right| \leq \mathcal{WB}\mathcal{P}_{H_b}(\hat{\sigma}, \hat{\omega}) \sqrt{|J|_{\hat{\omega}} |I|_{\hat{\sigma}}},$$

with $\mathcal{WB}\mathcal{P}_{H_b}(\hat{\sigma}, \hat{\omega}) < \infty$, for all I, J with $J \subset 3I$ and $I \subset 3J$. This is accomplished by showing that H_b satisfies the following inequality for any weight pair (σ, ω) :

$$\mathcal{WB}\mathcal{P}_{H_b}(\sigma, \omega) \leq \min \{ \mathfrak{T}_{H_b}(\sigma, \omega), \mathfrak{T}_{H_b}^*(\sigma, \omega) \} + C \mathcal{A}_2(\sigma, \omega).$$

This last inequality is proved for the Hilbert transform H in Proposition 2.9 of [LaSaUr2], and the same proof, which uses the two weight Hardy inequalities of Muckenhoupt, applies here with little change upon noting that $K_b(x) \approx \frac{1}{x}$ in the Hardy inequalities.

5.5. The energy conditions. We show in the first two subsubsections that the backward energy condition holds for the weight pair $(\hat{\sigma}, \hat{\omega})$, but not for the weight pair $(\hat{\sigma}, \hat{\omega})$. Then we show in the third subsubsection that the forward energy condition holds for the weight pair $(\hat{\sigma}, \hat{\omega})$. These proofs for the energy conditions follow the corresponding proofs for the energy conditions in [LaSaUr2], but are complicated by the estimates for the redistributed measure $\hat{\omega}$ and the reweighted measure $\hat{\sigma}$.

5.5.1. The backward Energy Condition for $(\hat{\sigma}, \hat{\omega})$. First we show that the backward Energy Condition for the weight pair $(\hat{\sigma}, \hat{\omega})$ holds:

$$(5.17) \quad \sum_{r=1}^{\infty} |I_r|_{\hat{\sigma}} \mathbb{E} \left(I_r, \hat{\sigma} \right)^2 \mathbb{P} \left(I_r, \mathbf{1}_{I_0} \hat{\omega} \right)^2 \leq (\mathcal{E}^*)^2 |I_0|_{\hat{\omega}}.$$

We use the following estimate, which shows that with the energy factor included, we obtain a strengthening of the A_2 condition.

Proposition 1. *For any interval $I \subset [0, 1]$, we have the inequality*

$$(5.18) \quad |I|_{\hat{\sigma}} \mathbb{E} \left(I, \hat{\sigma} \right)^2 \mathbb{P} \left(I, \hat{\omega} \right)^2 \lesssim |I|_{\hat{\omega}}.$$

Proof. We may assume that $\mathbb{E}(I; \sigma) \neq 0$. Let k be the smallest integer for which there is an r with $\dot{z}_r^k \in I$. Let n be the smallest integer so that for some s we have $\dot{z}_s^{k+n} \in I$ and $\dot{z}_s^{k+n} \neq \dot{z}_r^k$. We can estimate $\mathbb{E}(I; \hat{\sigma})$ in terms of n by

$$(5.19) \quad \mathbb{E} \left(I, \hat{\sigma} \right)^2 \lesssim \left(\frac{2}{N^2(1-\eta)} \right)^n.$$

Indeed, the worst case is when s is not unique. Then there are two choices of s – but not more. Let $\dot{z}_{s'}^{k+n} \in I$, where $s' \neq s$. Note that we then have

$$\frac{|I - \{\dot{z}_r^k\}|_{\hat{\sigma}}}{|I|_{\hat{\sigma}}} \lesssim \frac{\max \{\hat{s}_s^{k+n}, \hat{s}_{s'}^{k+n}\}}{\hat{s}_r^k} \lesssim \left(\frac{2}{N^2(1-\eta)} \right)^n,$$

and this gives (5.19) upon using the characterization of Energy as a variance:

$$\mathbb{E} \left(I, \hat{\sigma} \right)^2 \leq \frac{1}{|I|_{\hat{\sigma}}} \int_I \left| \frac{x - \dot{z}_r^k}{|I|} \right|^2 \hat{\sigma}(x) \leq \frac{1}{|I|_{\hat{\sigma}}} \int_{I \setminus \{\dot{z}_r^k\}} \hat{\sigma}.$$

Next we note from (5.5) that $|I|_{\hat{\sigma}} \approx |I_r^k|_{\hat{\sigma}}$ and $|I| \approx |I_r^k|$ and $\frac{|I_r^k|_{\hat{\sigma}}}{|I_r^k|} \approx \frac{1}{\kappa_r^k} \left(\frac{2}{N} \right)^k$, $\frac{|I|_{\hat{\omega}}}{|I_m^{k+n}|} \geq \frac{|I_m^{k+n}|_{\hat{\omega}}}{|I_m^{k+n}|} \approx \kappa_m^{k+n} \left(\frac{N}{2} \right)^{k+n}$ for some m so that $I \supset I_m^{k+n}$, and $\mathbb{P}(I, \hat{\omega}) \approx \mathbb{P}(I_r^k, \hat{\omega}) \simeq \kappa_r^k \left(\frac{N}{2} \right)^k$ where $\kappa_r^\ell \equiv \left(1 + \frac{1}{N} \right)^{H(I_r^\ell)} \left(1 - \frac{1}{N} \right)^{T(I_r^\ell)}$. Thus we have estimates for all of the factors on both sides of (5.18), and the following inequality is sufficient for (5.18):

$$\left[|I_r^k| \frac{1}{\kappa_r^k} \left(\frac{2}{N} \right)^k \right] \left[\left(\frac{2}{N^2(1-\eta)} \right)^n \right] \left[\left(\kappa_r^k \left(\frac{N}{2} \right)^k \right)^2 \right] \lesssim |I_m^{k+n}| \kappa_m^{k+n} \left(\frac{N}{2} \right)^{k+n}.$$

But this reduces to

$$\left(\frac{2}{N} \right)^{2n} \left(\frac{1}{1-\eta} \right)^n \lesssim \frac{\kappa_m^{k+n}}{\kappa_r^k} = \frac{\left(1 + \frac{1}{N} \right)^{H(I_m^{k+n})} \left(1 - \frac{1}{N} \right)^{T(I_m^{k+n})}}{\left(1 + \frac{1}{N} \right)^{H(I_r^k)} \left(1 - \frac{1}{N} \right)^{T(I_r^k)}},$$

which is in fact true for all pairs of n and k since I can only intersect the interval I_r^k among those in \mathcal{D} at generation k , and thus $I_m^{k+n} \subset I_r^k$, so that $T(I_m^{k+n}) - T(I_r^k) \leq n$ and $H(I_m^{k+n}) - H(I_r^k) \leq n$. Indeed, we then get

$$\frac{\left(1 + \frac{1}{N}\right)^{H(I_m^{k+n})} \left(1 - \frac{1}{N}\right)^{T(I_m^{k+n})}}{\left(1 + \frac{1}{N}\right)^{H(I_r^k)} \left(1 - \frac{1}{N}\right)^{T(I_r^k)}} \geq \left(\frac{1 - \frac{1}{N}}{1 + \frac{1}{N}}\right)^n = \left(\frac{N-1}{N+1}\right)^n \gtrsim \left(\frac{4}{N^2(1-\eta)}\right)^n,$$

for all $n \geq 1$ provided $\frac{N-1}{N+1} \geq \frac{4}{N^2(1-\eta)}$, e.g. if $N \geq 4$. \square

It is now clear that the pair of weights $(\hat{\sigma}, \hat{\omega})$ satisfy the backward Energy Condition. Indeed, let $I_0 \subset [0, 1]$ and let $\{I_r : r \geq 1\}$ be any partition of I_0 . From (5.18) we have

$$\sum_{r \geq 1} |I_r|_{\hat{\sigma}} \mathbb{E} \left(I_r, \hat{\sigma} \right)^2 \mathbb{P} \left(I_r, \hat{\omega} \right)^2 \lesssim \sum_{r \geq 1} |I_r|_{\hat{\omega}} = |I_0|_{\hat{\omega}}.$$

We will next see that the backward energy condition **fails** for the weight pair $(\hat{\sigma}, \hat{\omega})$ in which $\hat{\sigma}$ is no longer a sum of point masses.

5.5.2. Failure of the backward Energy Condition for $(\hat{\sigma}, \hat{\omega})$. We will choose a subdecomposition of $[0, 1]$ in which the backward Energy Condition for the weight pair $(\hat{\sigma}, \hat{\omega})$ is the same as the backward Pivotal Condition for the weight pair $(\hat{\sigma}, \hat{\omega})$ because the measure $\hat{\sigma}$ has energy equal to the ‘pivotal energy’ of σ , namely $\mathbb{E}(G_j^k, \hat{\sigma}) \approx \mathbb{E}(L_j^k, \hat{\sigma}) \approx 1$. Since the backward Pivotal Condition fails for the weight pair $(\hat{\sigma}, \hat{\omega})$, it follows that the backward Energy Condition fails as well for the weight pair $(\hat{\sigma}, \hat{\omega})$. Here are the details. We have from (5.5) that

$$\mathbb{P}(G_r^\ell, \hat{\omega}) \approx \mathbb{P}(I_r^\ell, \hat{\omega}) \approx \frac{|I_r^\ell|_{\hat{\omega}}}{|I_r^\ell|}, \quad \text{for all } r.$$

Thus for the decomposition $\bigcup_{\ell, r} G_r^\ell \subset [0, 1]$ we have

$$\begin{aligned} \sum_{\ell, r} |G_r^\ell|_{\hat{\sigma}} \mathbb{E}(G_r^\ell, \hat{\sigma})^2 \mathbb{P}(G_r^\ell, \hat{\omega})^2 &\approx \sum_{\ell, r} |G_r^\ell|_{\hat{\sigma}} \mathbb{P}(G_r^\ell, \hat{\omega})^2 \\ &\approx \sum_{\ell=0}^{\infty} \sum_r \frac{|G_r^\ell|_{\hat{\sigma}} |I_r^\ell|_{\hat{\omega}}}{|I_r^\ell|^2} |I_r^\ell|_{\hat{\omega}} \approx \sum_{\ell=0}^{\infty} \sum_r |I_r^\ell|_{\hat{\omega}} = \sum_{\ell=0}^{\infty} |[0, 1]_{\hat{\omega}} = \infty. \end{aligned}$$

5.5.3. The forward Energy Condition. It remains to verify that the measure pair $(\hat{\sigma}, \hat{\omega})$ satisfies the forward Energy Condition. We will in fact establish the pivotal condition (1.7)

$$\sum_{r=1}^{\infty} |I_r|_{\hat{\omega}} \mathbb{P} \left(I_r, \mathbf{1}_{I_0} \hat{\sigma} \right)^2 \leq \mathcal{P}^2 |I_0|_{\hat{\sigma}},$$

which then implies that $\mathcal{E}(\hat{\sigma}, \hat{\omega}) < \infty$ since $\mathbb{E}(I_r, \mathbf{1}_{I_0} \hat{\sigma})$. For this it suffices to show that the forward maximal inequality

$$(5.20) \quad \int M(f \hat{\sigma})^2 d\hat{\omega} \leq C \int |f|^2 d\hat{\sigma}$$

holds for the pair $(\hat{\sigma}, \hat{\omega})$. Indeed, as is well known, the Poisson integral is an ℓ^1 average of a sequence of expanding $\hat{\sigma}$ averages, and thus it is clearly dominated by the supremum of these averages, and in particular,

$$\mathbb{P} \left(I_r, \mathbf{1}_{I_0} \hat{\sigma} \right) \lesssim \inf_{x \in I_r} M \left(\mathbf{1}_{I_0} \hat{\sigma} \right).$$

Then we have

$$\sum_{r=1}^{\infty} |I_r|_{\hat{\omega}} \mathbb{P} \left(I_r, \mathbf{1}_{I_0} \hat{\sigma} \right)^2 \lesssim \sum_{r=1}^{\infty} |I_r|_{\hat{\omega}} \inf_{x \in I_r} M \left(\mathbf{1}_{I_0} \hat{\sigma} \right)^2 \leq \int M \left(\mathbf{1}_{I_0} \hat{\sigma} \right)^2 d\hat{\omega} \leq C \int \mathbf{1}_{I_0} d\hat{\sigma},$$

by (5.20) with $f = \mathbf{1}_{I_0}$.

Now (5.20) in turn follows from the testing condition

$$(5.21) \quad \int_Q M\left(\mathbf{1}_Q \hat{\sigma}\right)^2 d\hat{\omega} \leq C \int_Q d\hat{\sigma},$$

for all intervals Q (see [Saw1]). We will show (5.21) when $Q = I_r^\ell$, the remaining cases being an easy consequence of this one. Indeed, let \hat{z}_r^ℓ be the point in Q with smallest ℓ . Inequality (5.21) then remains unaffected by replacing Q with $Q \cap I_r^\ell$. Next we increase $Q \cap I_r^\ell$ to I_r^ℓ and note the left hand side of (5.21) increases while the right hand side remains comparable. Thus we have reduced matters to checking (5.21) when $Q = I_r^\ell$.

For the case $Q = I_r^\ell$ of (5.21), we use the fact from (5.5) that

$$(5.22) \quad \mathcal{M}\left(\mathbf{1}_{I_r^\ell} \hat{\sigma}\right)(x) \leq C \frac{1}{\kappa_r^\ell} \left(\frac{2}{N}\right)^\ell, \quad x \in \mathbf{E}^{(N)} \cap I_r^\ell,$$

where $\kappa_r^\ell \equiv \left(1 + \frac{1}{N}\right)^{H(I_r^\ell)} \left(1 - \frac{1}{N}\right)^{T(I_r^\ell)}$ as in (5.6). To see (5.22), note that for each $x \in I_r^\ell$ that also lies in the Cantor set $\mathbf{E}^{(N)}$, we have

$$\mathcal{M}\left(\mathbf{1}_{I_r^\ell} \hat{\sigma}\right)(x) \lesssim \sup_{(k,j): x \in I_j^k} \frac{1}{|I_j^k|} \int_{I_j^k \cap I_r^\ell} d\hat{\sigma} \approx \sup_{(k,j): x \in I_j^k} \frac{1}{\kappa_j^k} \frac{\left(\frac{1}{N}\right)^{k \vee \ell} \left(\frac{2}{N}\right)^{k \vee \ell}}{\left(\frac{1}{N}\right)^k} \approx \frac{1}{\kappa_r^\ell} \left(\frac{2}{N}\right)^\ell.$$

Now we consider for each fixed m , the approximations $\hat{\omega}^{(m)}$ and $\hat{\sigma}^{(m)}$ to the measures $\hat{\omega}$ and $\hat{\sigma}$ given in (5.13). For these approximations we have in the same way the estimate

$$\mathcal{M}\left(\mathbf{1}_{I_r^\ell} \hat{\sigma}^{(m)}\right)(x) \leq C \frac{1}{\kappa_r^\ell} \left(\frac{2}{N}\right)^\ell, \quad x \in \bigcup_{i=1}^{2^m} I_i^m.$$

Thus for each $m \geq 1$ we have

$$\begin{aligned} \int_{I_r^\ell} \mathcal{M}\left(\mathbf{1}_{I_r^\ell} \hat{\sigma}^{(m)}\right)^2 d\hat{\omega}^{(m)} &\leq C \sum_{i: I_i^m \subset I_r^\ell} \left(\frac{1}{\kappa_r^\ell}\right)^2 \left(\frac{2}{N}\right)^{2\ell} \kappa_i^m 2^{-m} \\ &\leq C 2^{m-\ell} \frac{1}{\kappa_r^\ell} \left(\frac{2}{N}\right)^{2\ell} 2^{-m} = C \hat{s}_r^\ell \approx C \int_{I_r^\ell} d\hat{\sigma}. \end{aligned}$$

Taking the limit as $m \rightarrow \infty$ yields the case $Q = I_r^\ell$ of (5.21). Indeed, from the Monotone Convergence Theorem we have

$$\int_{I_r^\ell} \mathcal{M}\left(\mathbf{1}_{I_r^\ell} \hat{\sigma}\right)^2 d\hat{\omega} = \lim_{n \rightarrow \infty} \int_{I_r^\ell} \mathcal{M}\left(\mathbf{1}_{I_r^\ell} \hat{\sigma}^{(n)}\right)^2 d\hat{\omega}.$$

From the continuity of $\mathcal{M}\left(\mathbf{1}_{I_r^\ell} \hat{\sigma}^{(n)}\right)^2$ on the support of $d\hat{\omega}^{(2n)}$, and the fact that the supports of $\hat{\omega}^{(m)}$ are decreasing, we have

$$\int_{I_r^\ell} \mathcal{M}\left(\mathbf{1}_{I_r^\ell} \hat{\sigma}^{(n)}\right)^2 d\hat{\omega} = \lim_{m \rightarrow \infty} \int_{I_r^\ell} \mathcal{M}\left(\mathbf{1}_{I_r^\ell} \hat{\sigma}^{(n)}\right)^2 d\hat{\omega}^{(m)}.$$

For $m \geq n$, we have

$$\int_{I_r^\ell} \mathcal{M}\left(\mathbf{1}_{I_r^\ell} \hat{\sigma}^{(n)}\right)^2 d\hat{\omega}^{(m)} \leq \int_{I_r^\ell} \mathcal{M}\left(\mathbf{1}_{I_r^\ell} \hat{\sigma}^{(m)}\right)^2 d\hat{\omega}^{(m)} \leq C \int_{I_r^\ell} d\hat{\sigma}$$

by monotonicity, and so altogether

$$\int_{I_r^\ell} \mathcal{M}\left(\mathbf{1}_{I_r^\ell} \hat{\sigma}\right)^2 d\hat{\omega} = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{I_r^\ell} \mathcal{M}\left(\mathbf{1}_{I_r^\ell} \hat{\sigma}^{(n)}\right)^2 d\hat{\omega}^{(m)} \leq C \int_{I_r^\ell} d\hat{\sigma}.$$

This completes our proof of the pivotal condition, and hence also the forward Energy Condition for the weight pair $(\hat{\sigma}, \hat{\omega})$.

5.6. The norm inequality. Here we show that the norm inequality for H_b holds with respect to the weight pair $(\hat{\sigma}, \hat{\omega})$. We first observe that we have already established above the following facts for the other weight pair $(\hat{\sigma}, \hat{\omega})$.

(1) The Muckenhoupt/NTV condition \mathcal{A}_2 holds:

$$\sup_I P(I, \hat{\omega}) \cdot P(I, \hat{\sigma}) = \mathcal{A}_2 < \infty.$$

(2) The forward testing condition holds:

$$\int_I |H_b(\mathbf{1}_I \hat{\sigma})|^2 d\hat{\omega} \lesssim |I|_{\hat{\sigma}}.$$

(3) The backward testing condition holds:

$$\int_I |H_b(\mathbf{1}_I \hat{\omega})|^2 d\hat{\sigma} \lesssim |I|_{\hat{\omega}}.$$

(4) The weak boundedness property holds:

$$\left| \int_J H_b(\mathbf{1}_I \hat{\sigma}) d\hat{\omega} \right| \lesssim \sqrt{|J|_{\hat{\omega}} |I|_{\hat{\sigma}}}.$$

(5) The forward energy condition holds:

$$\sum_J \left(\frac{P(J, \mathbf{1}_I \hat{\sigma})}{|J|} \right)^2 \|P_J^{\hat{\omega}} x\|_{L^2(\hat{\omega})}^2 \lesssim |I|_{\hat{\sigma}}.$$

(6) The backward energy condition holds:

$$\sum_J \left(\frac{P(J, \mathbf{1}_I \hat{\omega})}{|J|} \right)^2 \|P_J^{\hat{\sigma}} x\|_{L^2(\hat{\sigma})}^2 \lesssim |I|_{\hat{\omega}}.$$

Now we can apply our $T1$ theorem with an energy side condition in [SaShUr7] (or see [SaShUr6] or [SaShUr9]) to obtain the dual norm inequality

$$\int |H_b(g\hat{\omega})|^2 d\hat{\sigma} \lesssim \int |g|^2 d\hat{\omega}.$$

Consider the weight pair $(\hat{\sigma}, \hat{\omega})$. Since $H_b(g\hat{\omega})$ is constant on each interval L_j^k , we see that $\int |H_b(g\hat{\omega})|^2 d\hat{\sigma} = \int |H_b(g\hat{\omega})|^2 d\hat{\sigma}$, and hence the dual norm inequality for H_b holds with respect to the weight pair $(\hat{\sigma}, \hat{\omega})$.

This completes the proof of Theorem 1.

6. ENERGY REVERSAL AND THE $T1$ THEOREM

Here we prove Theorem 2 by first establishing reversal of energy and the necessity of the energy conditions, and then applying the main result from [SaShUr9]. So suppose our kernel $K(x, y)$ is gradient elliptic, i.e. satisfies

$$\left| \frac{\partial}{\partial x} K(x, y) \right|, \left| \frac{\partial}{\partial y} K(x, y) \right| \geq \frac{c}{(x-y)^2}$$

(the Hilbert kernel $K(x, y) = \frac{1}{y-x}$ satisfies $\frac{\partial}{\partial x} K(x, y) = \frac{1}{(x-y)^2}$). Then for a positive measure μ supported outside an interval J , and for $x, z \in J$ with $x > z$, we have

$$\begin{aligned} T\mu(x) - T\mu(z) &= \int_{\mathbb{R}} [K(x, y) - K(z, y)] d\mu(y) = \int_{\mathbb{R}} \left[\int_z^x \frac{\partial}{\partial t} K(t, y) dt \right] d\mu(y) \\ &\geq \int_{\mathbb{R}} \left[\int_z^x \frac{c}{(t-y)^2} dt \right] d\mu(y) = c(x-z) \int_{\mathbb{R}} \frac{d\mu(y)}{(x-y)(z-y)} \\ &\geq \frac{c}{4}(x-z) \int_{\mathbb{R}} \frac{d\mu(y)}{(c_J - y)^2} \approx \frac{c}{4}(x-z) \frac{P(J, \mu)}{|J|}, \end{aligned}$$

and hence

$$\begin{aligned}
& \int_J \int_J |T\mu(x) - T\mu(z)|^2 d\omega(x) d\omega(z) \\
& \geq \frac{c^2}{16} \left(\frac{P(J, \mu)}{|J|} \right)^2 \int_J \int_J |x - z|^2 d\omega(x) d\omega(z) \\
& = \frac{c^2}{8} \left(\frac{P(J, \mu)}{|J|} \right)^2 |J|_\omega \int_J |x - \mathbb{E}_L^\omega x|^2 d\omega(x) \\
& = \frac{c^2}{8} \left(\frac{P(J, \mu)}{|J|} \right)^2 |J|_\omega \|P_{J\mathbf{x}}^\omega\|_{L^2(\omega)}^2.
\end{aligned}$$

It now follows, using the definition of the punctured Muckenhoupt condition A_2^{punct} from [SaShUr9], that if

$\bigcup J_n \subset I$, then

$$\begin{aligned}
& \sum_n \left(\frac{P(J_n, \mathbf{1}_I \mu)}{|J_n|} \right)^2 \|P_{J_n \mathbf{x}}^\omega\|_{L^2(\omega)}^2 \\
& \lesssim \sum_n \left(\frac{P(J_n, \mathbf{1}_{J_n} \mu)}{|J_n|} \right)^2 \|P_{J_n \mathbf{x}}^\omega\|_{L^2(\omega)}^2 + \sum_n \left(\frac{P(J_n, \mathbf{1}_{I \setminus J_n} \mu)}{|J_n|} \right)^2 \|P_{J_n \mathbf{x}}^\omega\|_{L^2(\omega)}^2 \\
& \lesssim A_2^{\text{punct}} \sum_n |J_n|_\mu + \sum_n \frac{1}{|J_n|_\omega} \int_{J_n} \int_{J_n} |T(\mathbf{1}_{I \setminus J_n} \mu)(x) - T(\mathbf{1}_{I \setminus J_n} \mu)(z)|^2 d\omega(x) d\omega(z) \\
& \lesssim A_2^{\text{punct}} |I|_\mu + \sum_n \int_{J_n} |T(\mathbf{1}_{I \setminus J_n} \mu)(x)|^2 d\omega(x) \\
& \lesssim A_2^{\text{punct}} |I|_\mu + \sum_n \int_{J_n} |T(\mathbf{1}_{J_n} \mu)(x)|^2 d\omega(x) + \sum_n \int_{J_n} |T(\mathbf{1}_I \mu)(x)|^2 d\omega(x) \\
& \lesssim A_2^{\text{punct}} |I|_\mu + \sum_n \mathfrak{T} |J_n|_\mu + \mathfrak{T} |I|_\mu \lesssim (A_2^{\text{punct}} + \mathfrak{T}) |I|_\mu.
\end{aligned}$$

This shows that the energy conditions are controlled by the punctured Muckenhoupt conditions and the testing conditions, and now an application of the main theorem in [SaShUr9] completes the proof of Theorem 2.

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